"On a Method of Tracing Periodicities in a Series of Observations when the Periods are unknown." By VINAYEK NARAYEU NENE, First Assistant, Government Observatory, Colaba, Bombay. Communicated by C. CHAMBERS, F.R.S., Superintendent of the Observatory. Received May 30. Read June 15, 1882.

[PLATE 4.]

1. The object of this paper is to investigate a new method which will enable us to find out the period of an unknown inequality, and thence the inequality itself, from a series of observations—such as magnetical, meteorological, or astronomical—when they are of a periodic nature.

2. We assume that there is some large complex period, such that the observations—which need not necessarily correspond to equidistant times, but must have the time similarly divided in each general period—will be exactly repeated after such full period ;* but since we have not yet found such a period, although we have observations of thirty years' duration or more, say, of barometer, we may say that such a period is not yet completed, and we cannot, therefore, apply Bessel's method, which is applicable to only known periods, to find subordinate periodic inequalities. Diurnal and annual periods which have already been discovered may be considered as subordinate periods of the large period which we are considering. Our aim is first to ascertain the duration of subordinate periods one by one, and then to find out the amplitude for each period.

3. Any observations of a periodic nature may be represented by the equation—

$$Y=u+u_1\sin\left(U_1+\frac{2\pi x}{K_1}\right)+u_2\sin\left(U_2+\frac{2\pi x}{K_2}\right)+, \&c. \quad . \quad . \quad (1),$$

in which K_1, K_2, &c., which are all unknown periods which are in an ascending order of magnitude, but the relations of which with each other are also unknown ; Y is the phenomenon, and x the variable time on which the phenomenon depends ; u, u_1, u_2, and U_1, U_2, &c., are constants ; and 2π is of course equivalent to 360°.

4. The form given in Bessel's paper, embodied in the "Quarterly Weather Report of the Meteorological Department," Part IV, October to December, 1870, is—

$$Y=u+u_1\sin\left(U_1+\frac{2\pi x}{K}\right)+u_2\sin\left(U_2+\frac{4\pi x}{K}\right)+, \&c.,$$

which form can also be written—

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* Practically we shall deal only with observations that subdivide the large period spoken of into equal intervals.

$$Y=u+u_1\sin\left(U_1+\frac{2\pi x}{K}\right)+u_2\sin\left(U_2+\frac{2\pi x}{\frac{K}{2}}\right)+,\ \&c.\ \ .\ \ .\ \ (2),$$

in which K represents a period, after the completion of which the phenomenon Y constantly recurs, and x is the variable quantity on which the phenomenon depends. The given condition is that Y remains constant for values of x, which are increased or diminished by K, 2K, 3K, &c., u, u_1, u_2, and U_1, U_2, &c., being constants, and 2π equivalent to 360°.

5. Comparing the form of equation (1) and (2) it will be seen that they are both alike, except in subordinate periods which are in equation (1) in an ascending order of magnitude, and apparently unconnected with each other; and in equation (2) they are in a descending order of magnitude, being connected with a proportion $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c., in order.

If we suppose that the least common multiple of the subordinate periods K_1, K_2, K_3, &c., in equation (1) is the same as K in equation (2), then the two equations are identical except as to the inversion of the order of the terms, and in that some of the terms of (2) whose coefficients are zero will not appear in equation (1).

6. It might be asked what reasons make us give preference to the form of equation (1) instead of to the form of equation (2); the reply is that the form of equation (2) includes all the subordinate periods that are possible whether they may really exist in a particular series of observations or not, and it is therefore confounding and complicated on account of introducing unnecessary terms in the equation, whereas the form of equation (1) contains only all the unknown subordinate periods, that is, those which may really exist in a particular series of observations. To illustrate practically the advantage of the one over the other in point of simplicity, we shall take a case in which a particular set of observations is composed of subordinate periods of 5, 7, and 15 days only. This can be represented by the form of equation (1) by three terms only, while by the form of equation (2) 105 are required, although it will prove, after all the periods are found, that all of them have *zero* for their value except the terms whose periods are $\frac{1}{7}, \frac{1}{15}, \frac{1}{21}$ of the full period. On the other hand, if a particular set of observations is really composed of subordinate periods 105 in number, viz., those whose periods are 105, $\frac{105}{2}$, $\frac{105}{3}$, &c., the form of equation (1) will still hold good. It really does not affect the forms but the ideas only.

Having found the equation for any kind of observations of a periodic nature, our next step is to see what results take place if certain operations be performed on the observed values of equation (1).

7. Before doing this let us prove some preliminary propositions.

It has already been established in chapters on the subject of Summation in Trigonometry that the sum of n terms of the sines of angles which are in arithmetical progression from the first term of the series (3) written below is $u \sin \dfrac{\left(\mathrm{U}+\frac{n-1}{2}b\right)\sin\frac{nb}{2}}{\sin\frac{b}{2}}$, and that beginning at the second term it is $u \sin \left\{(\mathrm{U}+b)+\frac{n-1}{2}b\right\}\dfrac{\sin\frac{nb}{2}}{\sin\frac{b}{2}}$, and so on. The object of the following investigation is to find what relations exist between the series (3) and the series derived from it, in the manner that we are going to describe below.

Let the proposed series be

$$u\{\sin \mathrm{U}+\sin(\mathrm{U}+b)+\sin(\mathrm{U}+2b)+\sin(\mathrm{U}+3b)\ \ldots\ \&c.\} \quad . \quad (3),$$

from which we see that the mean of n terms from the first term of the series is $u \sin \dfrac{\left(\mathrm{U}+\frac{n-1}{2}b\right)\sin\frac{nb}{2}}{n\sin\frac{b}{2}}$, and that the mean of n terms from the second term of the series is—

$$\frac{u \sin\left\{(\mathrm{U}+b)+\frac{n-1}{2}b\right\}\sin\frac{nb}{2}}{n\sin\frac{b}{2}},$$

or, by reducing, we obtain

$$\frac{u \sin\left\{\mathrm{U}+\frac{n+1}{2}b\right\}\sin\frac{nb}{2}}{n\sin\frac{b}{2}},$$

and similarly the mean of n terms from the third term of the series is $u \sin\left(\mathrm{U}+\frac{n+3}{2}b\right)\dfrac{\sin nb}{n\sin\frac{b}{2}}$, and so on. Thus the mean series will be—

$$u\left\{\sin\left(\mathrm{U}+\frac{n-1}{2}b\right)\frac{\sin\frac{nb}{2}}{n\sin\frac{b}{2}}+\sin\left(\mathrm{U}+\frac{n+1}{2}b\right)\frac{\sin\frac{nb}{2}}{n\sin\frac{b}{2}}\right.$$

$$\left.+\sin\left(\mathrm{U}+\frac{n+3}{2}b\right)\frac{\sin\frac{nb}{2}}{n\sin\frac{b}{2}}+,\ \&c.\right\} \quad . \quad (4.)$$

Again, from this new series we see that the mean of the two terms from the first term of the mean series is $u\sin\left(\mathrm{U}+\frac{nb}{2}\right)\frac{\sin\frac{nb}{2}\cos\frac{b}{2}}{n\sin\frac{b}{2}}$, and that the mean of the two terms from the second term of the mean series is $u\sin\left(\mathrm{U}+\frac{n+2}{2}b\right)\frac{\sin\frac{nb}{2}\cos\frac{b}{2}}{n\sin\frac{b}{2}}$, and similarly the mean of the two terms from the third term of the mean series is

$$u\sin\left(\mathrm{U}+\frac{n+4}{2}b\right)\frac{\sin\frac{nb}{2}\cos\frac{b}{2}}{n\sin\frac{b}{2}},$$

and so on.

Thus the second mean series will be—

$$u\left\{\sin\left(\mathrm{U}+\frac{nb}{2}\right)\frac{\sin\frac{nb}{2}\cos\frac{b}{2}}{n\sin\frac{b}{2}}+\sin\left(\mathrm{U}+\frac{n+2}{2}b\right)\frac{\sin\frac{nb}{2}\cos\frac{b}{2}}{n\sin\frac{b}{2}}\right.$$

$$\left.+\sin\left(\mathrm{U}+\frac{n+4}{2}b\right)\frac{\sin\frac{nb}{2}\cos\frac{b}{2}}{n\sin\frac{b}{2}}+,\ \&c.\right\} \quad . \quad (5).$$

Now in these results we see that the following laws hold—

I. When n is an odd number the first factors of the terms of the first mean series are the same terms in order as the $\left(\frac{n+1}{2}\right)$th term of the proposed series.

II. When n is an even number the first factors of the terms of the second mean series are the same terms in order as the $\left(\frac{n+2}{2}\right)$th term of the proposed series.

III. The second factor in each term of the first or second mean series is a common factor to its own series.

IV. Each first factor of the terms of the first or second mean series is the middle term of the proposed series from which the corresponding mean term is derived.

Thus the investigation established, that the first mean series is proportional with the proposed series from $\left(\frac{n+1}{2}\right)$th term when n is an odd number, and that the second mean series is proportional with the proposed series from $\left(\frac{n+2}{2}\right)$th term when n is an even number.

8. Suppose in the proposed series that $b=\frac{2\pi}{n}$; then since

$$\sin\frac{nb}{2}=\sin\pi=0,$$

the mean of the sines of the proposed series will be *zero*, and consequently the first mean series or the second mean series will also be *zero*.

9. The factor $\frac{\sin\frac{nb}{2}}{n\sin\frac{b}{2}}$ or $\frac{\sin\frac{nb}{2}\cos\frac{b}{2}}{n\sin\frac{b}{2}}$ is a proper fraction.

Put $\frac{b}{2}=\text{A}$, thus the first factor becomes $\frac{\sin n\text{A}}{n\sin\text{A}}$.

Now $\sin n\text{A} = \sin\{\text{A}+(n-1)\text{A}\} = \sin(n-1)\text{A}\cos\text{A} + \cos(n-1)\text{A}$ $\sin\text{A}$, and $\sin(n-1)\text{A}\cos\text{A}=\sin(n-2)\text{A}\cos^2\text{A}+\cos(n-2)\text{A}\cos\text{A}$ $\sin\text{A}$; and, similarly,

$$\sin(n-2)\text{A}\cos^2\text{A}=\sin(n-3)\text{A}\cos^3\text{A}+\cos(n-3)\text{A}\cos^2\text{A}\sin\text{A}.$$

. . . .

$$\sin 3\text{A}\cos^{(n-3)}\text{A}=\sin 2\text{A}\cos^{(n-2)}\text{A}+\cos 2\text{A}\cos^{(n-3)}\text{A}\sin\text{A}.$$

$$\sin 2\text{A}\cos^{(n-2)}\text{A}=\sin\text{A}\cos^{(n-1)}\text{A}+\cos\text{A}\cos^{n-2}\text{A}\sin\text{A}.$$

Thus

$$\sin n\text{A}=\sin\text{A}\{\cos(n-1)\text{A}+\cos(n-2)\text{A}\cos\text{A}+\cos(n-3)\text{A}\cos^2\text{A}\ldots$$
$$\cos 2\text{A}\cos^{n-3}\text{A}+\cos\text{A}\cos^{n-2}\text{A}+\cos^{n-1}\text{A}\}.$$

The number of terms on the right hand side of the equation is n, and the numerical sum (regardless of sign) of the terms in the bracket is obviously less than n, thus the numerical value of $\sin n\text{A}$ is less than the numerical value of $n\sin\text{A}$; and since $\cos\text{A}$ is numerically less than *unity*, the other factor is also a proper fraction.

10. The following conditions will show without proof when the first and the second factors respectively will have the positive or negative sign :—

The first factor is positive when nb lies between $2\pi(2\text{N})$ and $2\pi(2\text{N}+1)$, and b lies between $2\pi(2\text{N}')$ and $2\pi(2\text{N}'+1)$, or when nb lies between $2\pi(2\text{N}+1)$ and $2\pi(2\text{N}+2)$, and b lies between $2\pi(2\text{N}'+1)$ and $2\pi(2\text{N}'+2)$.

The second factor is positive when nb lies between $2\pi(2N)$ and $2\pi(2N+1)$, and b lies between $\pi(2N')$ and $\pi(2N'+1)$, or when nb lies between $2\pi(2N+1)$ and $2\pi(2N+2)$, and b lies between $\pi(2N'+1)$ and $\pi(2N'+2)$.

The first factor is negative when nb lies between $2\pi(2N+1)$ and $2\pi(2N+2)$, and b lies between $2\pi(2N')$ and $2\pi(2N'+1)$, or when nb lies between $2\pi(2N)$ and $2\pi(2N+1)$, and b lies between $2\pi(2N'+1)$ and $2\pi(2N'+2)$,

The second factor is negative when nb lies between $2\pi(2N+1)$ and $2\pi(2N+2)$, and b lies between $\pi(2N')$ and $\pi(2N'+1)$, or when nb lies between $2\pi(2N)$ and $2\pi(2N+1)$, and b lies between $\pi(2N'+1)$ and $\pi(2N'+2)$, where N and N′ stand for *zero* or any integral number.

11. To trace the changes in the factor $\dfrac{\sin\frac{nb}{2}}{n\sin\frac{b}{2}}$ or $\dfrac{\sin\frac{nb}{2}\cos\frac{b}{2}}{\sin\frac{b}{2}}$ as the angle nb varies between 0 and 2π, between 2π and 4π, between 4π and 6π, &c. We shall discuss this subject in a particular case only, viz., when $b<2\pi$, as there is no necessity of discussing all the cases for our future purpose.

We have already shown that when $b=\dfrac{2\pi}{n}$, or when $nb=2\pi$ the factor has the value *zero*, and it is easy to see that when $nb=4\pi$ or $=6\pi$ or $=2N\pi$, where N is some whole number, the factor has also the value *zero*.

We have now to find what value of nb makes the factor a maximum when $nb<2\pi$. Put $\dfrac{nb}{2}=x$ and therefore $n=\dfrac{2x}{b}$, thus the first factor becomes—

$$\frac{\sin x}{\frac{2x}{b}\sin\frac{b}{2}}=\frac{b\sin x}{2x\sin\frac{b}{2}}:$$

let $$u'=\frac{b\sin x}{2x\sin\frac{b}{2}},\text{ remove constant factors,}$$

thus $$u=\frac{\sin x}{x};$$

then $$\frac{du}{dx}=\frac{x\cos x-\sin x}{x^2},$$

$$\frac{d^2u}{dx^2}=-\frac{\sin x}{x}-\frac{2\cos x}{x^2}+\frac{2\sin x}{x^3}.$$

If we put $\dfrac{du}{dx}=0$ we obtain $\tan x=x$, therefore $x=0$, and when $x=0$

we have

$$\frac{d^2u}{dx^2}=-\frac{\sin x}{x}-\frac{2\cos x}{x^2}+\frac{2\cos x}{3x^2},$$

$$=-\frac{\sin x}{x}-\frac{4\cos x}{3x^2}=-1-\infty.$$

Thus when $x=0$, $\frac{d^2u}{dx^2}$ is negative.

Therefore when $x=0$ u has a maximum value. Thus we see that as the angle nb changes from 0 to 2π, the factor diminishes from the limiting value *unity* to *zero*.

We shall next see when we shall have a minimum value of the factor when nb changes from 2π to 4π.

Put $\frac{nb}{2}=\pi+x$, and consequently $n=\frac{\pi+x}{\frac{b}{2}}$; thus the first factor becomes—

$$\frac{\sin(\pi+x)}{\frac{+x}{\frac{b}{2}}\sin\frac{b}{2}}=\frac{\frac{b}{2}\sin(\pi+x)}{(\pi+x)\sin\frac{b}{2}}=\frac{-\frac{b}{2}\sin x}{(\pi+x)\sin\frac{b}{2}}.$$

Let $$u'=\frac{-\frac{b}{2}\sin x}{(\pi+x)\sin\frac{b}{2}},$$ remove constant factors,

thus $$u=\frac{-\sin x}{\pi+x},$$

then $$\frac{du}{dx}=\frac{-(\pi+x)\cos x+\sin x}{(\pi+x)^2},$$

$$\frac{d^2u}{dx^2}=\frac{\sin x}{\pi+x}+\frac{2\cos x}{(\pi+x)^2}-\frac{2\sin x}{(\pi+x)^3}.$$

If we put $\frac{du}{dx}=0$ we obtain $\tan x=\pi+x$, therefore $x=\frac{77^\circ\cdot5}{57^\circ\cdot3}$.

Therefore when $x=0$,

$$\frac{d^2u}{dx^2}=\frac{\cdot97630}{4\cdot493}+\frac{2(\cdot21644)}{(4\cdot493)^2}-\frac{2(\cdot97630)}{(\cdot493)^3}=\text{plus something};$$

therefore when $x=\frac{77^\circ\cdot5}{57^\circ\cdot3}$, u has a minimum value.

Thus when $nb=2\left(\pi+\frac{77^\circ\cdot5}{57^\circ\cdot3}\right)$ we have a minimum value, and similarly we can show that when $nb=2\left(2\pi+\frac{82^\circ\cdot5}{57^\circ\cdot3}\right)$ we shall have another maximum, and when $nb=2\left(3\pi+\frac{\text{something between }82\cdot5\text{ and }90}{57\cdot3}\right)$ we shall have another minimum, and so on alternately; for the general equation from which these values are to be determined is $\tan x=\mathrm{N}\pi+x$, where N is *zero* or some integral number.

We have taken the first factor only in dealing with the above subject, but it will be seen by inspection that the same results would have followed if the second factor had been taken.

These changes of factors have been worked out for a particular case, and curved in fig. 1 (Plate 4) to give the reader a general idea of them.

12. Now we shall repeat the equation (1) and proceed as follows:—

$$\mathrm{Y}=u+u_1\sin\left(\mathrm{U}_1+\frac{2\pi\mathrm{K}}{\mathrm{K}_1}\right)+u_2\sin\left(\mathrm{U}_2+\frac{2\pi\mathrm{K}}{\mathrm{K}_2}\right)+u_3\sin\left(\mathrm{U}_3+\frac{2\pi\mathrm{K}}{3}\right)+,\&c.$$

Let c be a small period or interval say of an hour, a day, a week, or a month, &c., as the case may be, at which observations are available, then writing b_1 for $\frac{2\pi c}{\mathrm{K}_1}$; b_2 for $\frac{2\pi c}{\mathrm{K}_2}$; b_3 for $\frac{2\pi c}{\mathrm{K}_3}$, &c., in the above equation (and appropriately modifying the constants U_1, U_2, &c., if x and c are reckoned from different epochs), the successive observations will be equal to the sum of the terms of the following expressions—

1st obs. $=u+u_1\sin\mathrm{U}_1+u_2\sin\mathrm{U}_2+u_3\sin\mathrm{U}_3+$ &c.

2nd obs. $=u+u_1\sin(\mathrm{U}_1+b_1)+u_2\sin(\mathrm{U}_2+b_2)+u_3\sin(\mathrm{U}_3+b_3)+$, &c.

3rd obs. $=u+u_1\sin(\mathrm{U}_1+2b_1)+u_2\sin(\mathrm{U}_2+2b_2)+u_3\sin(\mathrm{U}_3+2b_3)+$, &c.

4th obs. $=u+u_1\sin(\mathrm{U}_1+3b_1)+u_2\sin(\mathrm{U}_2+3b_2)+u_3\sin(\mathrm{U}_3+3b_3)+$, &c.

. . . .

$(n+1)^{\text{th}}=u+u_1\sin(\mathrm{U}_1+nb_1)+u_2\sin(\mathrm{U}_2+nb_2)+u_3\sin(\mathrm{U}_3+nb_3)+$, &c.

In order to save room we shall henceforth use the factor $\frac{1}{q_1}$, $\frac{1}{q_2}$, $\frac{1}{q_3}$, &c., in place of $\dfrac{\sin\frac{mb_1}{2}}{m\sin\frac{b_1}{2}}$, $\dfrac{\sin\frac{mb_2}{2}}{m\sin\frac{b_2}{2}}$, $\dfrac{\sin\frac{mb_3}{2}}{m\sin\frac{b_3}{2}}$, &c., respectively.

13. Suppose we have a table ruled with horizontal and vertical lines; in the first horizontal line let us enter the numerical values of the

observations in order. Now let successive means be taken of m observations (m being an odd integer) advancing to the right by one step at each operation, and entering the respective means under the $\frac{m+1}{2}^{\text{th}}$, $\frac{m+3}{2}^{\text{th}}$, &c., terms of the observations, thus forming a second line of our table; the means in our second line taken in order will be equal to the sum of the terms of the following expressions—

$$\text{1st mean} = u + u_1 \sin\left(\mathrm{U}_1 + \frac{m-1}{2}b_1\right)\frac{1}{q_1} + u_2 \sin\left(\mathrm{U}_2 + \frac{m-1}{2}b_2\right)\frac{1}{q_2}$$
$$+ u_3 \sin\left(\mathrm{U}_3 + \frac{m-1}{2}b_3\right)\frac{1}{q_3} +, \text{ \&c.}$$

$$\text{2nd mean} = u + u_1 \sin\left(\mathrm{U}_1 + \frac{m+1}{2}b_1\right)\frac{1}{q_1} + u_2 \sin\left(\mathrm{U}_2 + \frac{m+1}{2}b_2\right)\frac{1}{q_2}$$
$$+ u_3 \sin\left(\mathrm{U}_3 + \frac{m+1}{2}b_3\right)\frac{1}{q_3} +, \text{ \&c.}$$

$$\text{3rd mean} = u + u_1 \sin\left(\mathrm{U}_1 + \frac{m+3}{2}b_1\right)\frac{1}{q_1} + u_2 \sin\left(\mathrm{U}_2 + \frac{m+3}{2}b_2\right)\frac{1}{q_2}$$
$$+ u_3 \sin\left(\mathrm{U}_3 + \frac{m+3}{2}b_3\right)\frac{1}{q_3} +, \text{ \&c.}$$

. . . .

$$\{n-(m-1)\}^{\text{th}} = u + u_1 \sin\left[\mathrm{U}_1 + \left\{n - \frac{m+1}{2}\right\}b_1\right]\frac{1}{q_1}$$
$$+ u_2 \sin\left[\mathrm{U}_2 + \left\{n - \frac{m+1}{2}\right\}b_2\right]\frac{1}{q_2} + u_3 \sin\left[\mathrm{U}_3 + \left\{n - \frac{m+1}{2}\right\}b_3\right]\frac{1}{q_3} +, \text{ \&c.}$$

If these means which are entered be subtracted from the observations column by column, then the remainders will severally be equal to the sum of the terms of the following expressions—

$$\text{1st remainder} = u_1 \sin\left(\mathrm{U}_1 + \frac{m-1}{2}b_1\right)\frac{q_1-1}{q_1} + u_2 \sin\left(\mathrm{U}_2 + \frac{m-1}{2}b_2\right)\frac{q_2-1}{q_2}$$
$$+ u_3 \sin\left(\mathrm{U}_3 + \frac{m-1}{2}b_3\right)\frac{q_3-1}{q_3} +, \text{ \&c.}$$

$$\text{2nd remainder} = u_1 \sin\left(\mathrm{U}_1 + \frac{m+1}{2}b_1\right)\frac{q_1-1}{q_1} + u_2 \sin\left(\mathrm{U}_2 + \frac{m+1}{2}b_2\right)\frac{q_2-1}{q_3}$$
$$+ u_3 \sin\left(\mathrm{U}_3 + \frac{m+1}{2}b_3\right)\frac{q_3-1}{q_3} +, \text{ \&c.}$$

$$\text{3rd remainder} = u_1 \sin\left(\mathrm{U}_1 + \frac{m+3}{2}b_1\right)\frac{q_1-1}{q_1}$$

$$+u_2 \sin\left(U_2+\frac{m+3}{2}b_2\right)\frac{q_2-1}{q_2}+u_3 \sin\left(U_3+\frac{m+3}{2}b_3\right)\frac{q_3-1}{q_3}+, \&c.$$

. . . .

$$\{n-(m-1)\}^{\text{th}}=u_1 \sin\left\{U_1+\left(n-\frac{m+1}{2}\right)b_1\right\}\frac{q_1-1}{q_1}$$

$$+u_2 \sin\left\{U_2+\left(n-\frac{m+1}{2}\right)b_2\right\}\frac{q_2-1}{q_2}+u_3 \sin\left\{U_3+\left(n-\frac{m+1}{2}\right)b_3\right\}\frac{q_3-1}{q_3}$$
$$+, \&c.$$

Let these remainders be entered in the third line of our table.

Again, if successive means be taken of m of these remainders precisely in the same manner as the means of the observations which were taken before, we shall have a set of second means, and similarly if these means be subtracted from the first set of remainders we shall have a second set of remainders; and, further, if these operations be continued r times we shall have an rth set of means and an rth set of remainders, and the rth set of means taken in order will be equal to the sum of the terms of the following expressions*—

$$\text{1st mean}=u_1 \sin\left(U_1+\frac{m-1}{2}rb_1\right)\frac{(q_1-1)^{r-1}}{q_1^r}$$
$$+u_2 \sin\left(U_2+\frac{m-1}{2}rb_2\right)\frac{(q_2-1)^{r-1}}{q_2^r}+, \&c.$$

$$\text{2nd mean}=u_1 \sin\left(U_1+\frac{m-1}{2}rb_1+b_1\right)\frac{(q_1-1)^{r-1}}{q_1^r}$$
$$+u_2 \sin\left(U_2+\frac{m-1}{2}rb_2+b_2\right)\frac{(q_2-1)^{r-1}}{q_2^r}+, \&c.$$

$$\text{3rd mean}=u_1 \sin\left(U_1+\frac{m-1}{2}rb_1+2b_1\right)\frac{(q_1-1)^{r-1}}{q_1^r}$$
$$+u_2 \sin\left(U_2+\frac{m-1}{2}rb_2+2b_2\right)\frac{(q_2-1)^{r-1}}{q_2^r}+, \&c.$$

. . . .

$$\left\{n-\left(m-1\right)r\right\}^{\text{th}}=u_1 \sin\left\{U_1+\left(n-\frac{m-1}{2}r-1\right)b_1\right\}\frac{(q_1-1)^{r-1}}{q_1^r}$$
$$+u_2 \sin\left\{U_2+\left(n-\frac{m-1}{2}r-1\right)b_2\right\}\frac{(q_2-1)^{r-1}}{q_2^r}+, \&c.$$

And similarly each of the rth remainders will be represented to be equal to the sum of the terms of the following expressions—

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* See two next paragraphs.

$$\text{1st remainder} = u_1 \sin\left(\mathrm{U}_1 + \frac{m-1}{2} r b_1\right) \frac{(q_1-1)^r}{q_1^r}$$
$$+ u_2 \sin\left(\mathrm{U}_2 + \frac{m-1}{2} r b_2\right) \frac{(q_2-1)^r}{q_2^r} +, \text{\&c.}$$

$$\text{2nd remainder} = u_1 \sin\left(\mathrm{U}_1 + \frac{m-1}{2} r b_1 + b_1\right) \frac{(q_1-1)^r}{q_1^r}$$
$$+ u_2 \sin\left(\mathrm{U}_2 + \frac{m-1}{2} r b_2 + b_2\right) \frac{(q_2-1)^r}{q_2^r} +, \text{\&c.}$$

$$\text{3rd remainder} = u_1 \sin\left(\mathrm{U}_1 + \frac{m-1}{2} r b_1 + 2b_1\right) \frac{(q_1-1)^r}{q_1^r}$$
$$+ u_2 \sin\left(\mathrm{U}_2 + \frac{m-1}{2} r b_2 + 2b_2\right) \frac{(q_2-1)^r}{q_2^r} +, \text{\&c.}$$

. . . .

$$\{n-(m-1)r\}^{\text{th}} = u_1 \sin\left\{\mathrm{U}_1 + \left(n - \frac{m-1}{2} r - 1\right) b_1\right\} \frac{(q_1-1)^r}{q_1^r}$$
$$+ u_2 \sin\left\{\mathrm{U}_2 + \left(n - \frac{m-1}{2} r - 1\right) b_2\right\} \frac{(q_2-1)^r}{q_2^r} +, \text{\&c.}$$

Pursuing the mode of entry adopted for the first means and first remainders, the rth means will occupy the $2r$th line of our table and the rth remainders the $(2r+1)$th line.

In future we shall for shortness write—

$$\mathrm{U}_1 + \frac{m-1}{2} r b_1 = \mathrm{V}_1; \quad \mathrm{U}_2 + \frac{m-1}{2} r b_2 = \mathrm{V}_2, \text{\&c.}$$

14. The following calculations are made for ready reference, and to show how the factors of the form $\frac{(q-1)^{r-1}}{q^r}$ and $\frac{(q-1)^r}{q^r}$ which occurred in our rth means and rth remainders are obtained—

The factor for—

1st means $= \frac{1}{q}$.

1st remainders $= 1 - \frac{1}{q} = \frac{q-1}{q}$.

2nd means $= \frac{q-1}{q^2}$.

$$\text{2nd remainders} = \frac{q-1}{q} - \frac{q-1}{q^2} = \frac{(q-1)q-(q-1)}{q^2} = \frac{(q-1)(q-1)}{q^2}$$
$$= \frac{(q-1)^2}{q^2}.$$

3rd means $=\dfrac{(q-1)^2}{q^3}$.

3rd remainders $=\dfrac{(q-1)^2}{q^2}-\dfrac{(q-1)^2}{q^3}=\dfrac{(q-1)^2q-(q-1)^2}{q^3}$

$$=\frac{(q-1)^2(q-1)}{q^3}=\frac{(q-1)^3}{q^3}.$$

. . . .

rth means $=\dfrac{(q-1)^{r-1}}{q^r}$.

rth remainders $=\dfrac{(q-1)^{r-1}}{q^{r-1}}-\dfrac{(q-1)^{r-1}}{q^r}=\dfrac{(q-1)^{r-1}q-(q-1)^{r-1}}{q^r}$

$$=\frac{(q-1)^{r-1}(q-1)}{q^r}=\frac{(q-1)^r}{q^r}.$$

15. We must notice that at each operation of taking means we lose $\dfrac{m-1}{2}$ values at each end of the series, thus the rth mean series commences in our form after $\dfrac{m-1}{2}r$ observations. For this reason suppose strong vertical lines are drawn in our form at the $\dfrac{(m-1)}{2}r^{\text{th}}$ observation from each end, and let us confine our attention hereafter to the series embraced between the two strong vertical lines.

16. Then, of the numbers occupying the first vertical column from the first of the strong vertical lines mentioned above, the following, viz., an observation, an individual of the first set of means, an individual of the second set of means, an individual of the third set of means, &c., an individual of the rth set of means, and an individual of the rth set of remainders, will have for their mathematical equivalents the following expressions, taken in order, viz.,

1st observation from the observation series

$$=u+u_1\sin(\text{V}_1)+u_2\sin(\text{V}_2)+,\ \&\text{c}.$$

1st mean from the first set of means

$$=u+u_1\sin(\text{V}_1)\frac{1}{q_1}+u_2\sin(\text{V}_2)\frac{1}{q_2}+,\ \&\text{c}.$$

1st mean from the second set of means

$$=u_1\sin(\text{V}_1)\frac{(q_1-1)}{q_1{}^2}+u_2\sin(\text{V}_2)\frac{(q_2-1)}{q_2{}^2}+,\ \&\text{c}.$$

1st mean from the third set of means

$$=u_1 \sin(V_1)\frac{(q_1-1)^2}{q_1^3}+u_2 \sin(V_2)\frac{(q_2-1)^2}{q_2^3}+, \text{ \&c.}$$

. . . .

1st mean from the rth set of means

$$=u_1 \sin(V_1)\frac{(q_1-1)^{r-1}}{q_1^r}+u_2 \sin(V_2)\frac{(q_2-1)^{r-1}}{q_2^r}+, \text{ \&c.}$$

1st remainder from rth set of remainder

$$=u_1 \sin(V_1)\frac{(q_1-1)^r}{q_1^r}+u_2 \sin(V_2)\frac{(q_2-1)^r}{q_2^r},$$

and similarly the corresponding numbers from the second vertical column will be represented by the following expressions—

2nd observation from the observation series

$$=u+u_1 \sin(V_1+b_1)+u_2 \sin(V_2+b_2)+, \text{ \&c.}$$

2nd mean from the first set of means

$$=u+u_1 \sin(V_1+b_1)\frac{1}{q_1}+u_2 \sin(V_2+b_2)\frac{1}{q_2}+, \text{ \&c.}$$

2nd mean from the second set of means

$$=u_1 \sin(V_1+b_1)\frac{q_1-1}{q_1^2}+u_2 \sin(V_2+b_2)\frac{(q_2-1)}{q_2^2}+, \text{ \&c.}$$

2nd mean from the third set of means

$$=u_1 \sin(V_1+b_1)\frac{(q_1-1)^2}{q_1^3}+u_2 \sin(V_2+b_2)\frac{(q_2-1)^2}{q_2^3}+, \text{ \&c.}$$

. . . .

2nd mean from the rth set of means

$$=u_1 \sin(V_1+b_1)\frac{(q_1-1)^{r-1}}{q_1^r}+u_2 \sin(V_2+b_2)\frac{(q_2-1)^{r-1}}{q_2^r}+, \text{ \&c.}$$

2nd remainder from the rth set of remainders

$$=u_1 \sin(V_1+b_1)\frac{(q_1-1)^r}{q_1^r}+u_2 \sin(V_2+b_2)\frac{(q_2-1)^r}{q_2^r}+, \text{ \&c.}$$

And the corresponding numbers from the $(p+1)$th vertical column will be represented by the following expressions—

$(p+1)$th observation from the observation series

$$=u+u_1 \sin(V_1+pb_1)+u_2 \sin(V_2+pb_2)+, \text{ \&c.}$$

$(p+1)$th mean from the first set of means

$$=u+u_1 \sin(V_1+pb_1)\frac{1}{q_1}+u_2 \sin(V_2+pb_2)\frac{1}{q_2}+, \text{ \&c.}$$

$(p+1)$th mean from the second set of means

$$=u_1 \sin(\mathrm{V}_1+pb_1)\frac{q_1-1}{q_1^2}+u_2 \sin(\mathrm{V}_2+pb_2)\frac{q_2-1}{q_2^2}+, \text{ \&c.}$$

$(p+1)$th mean from the third set of means

$$=u_1 \sin(\mathrm{V}_1+pb_1)\frac{(q_1-1)^2}{q_1^3}+u_2 \sin(\mathrm{V}_2+pb_2)\frac{(q_2-1)^2}{q_2^3}+, \text{ \&c.}$$

. . . .

$(p+1)$th mean from the rth set of means

$$=u_1 \sin(\mathrm{V}_1+pb_1)\frac{(q_1-1)^{r-1}}{q_1^r}+u_2 \sin(\mathrm{V}_2+pb_2)\frac{(q_2-1)^{r-1}}{q_2^r}+, \text{ \&c.}$$

$(p+1)$th remainder from the rth set of remainders

$$=u_1 \sin(\mathrm{V}_1+pb_1)\frac{(q_1-1)^r}{q_1^r}+u_2 \sin(\mathrm{V}_2+pb_2)\frac{(q_2-1)^{r-1}}{q_2^r}+, \text{ \&c.}$$

And therefore if we use a variable integral number P (which may be zero) as a multiple to the angle of the type b, we can make one general expression serve to represent the numbers of any one horizontal line; this will be for the 1st, 3rd, 5th, &c., lines as follows :—

1st line, observation series

$$=u+\{u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)\}+\{u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\}+, \text{ \&c.}$$

3rd line, 1st mean series

$$=u+\{u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)\}\frac{1}{q_1}+\{u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\}\frac{1}{q_2}+, \text{ \&c.}$$

5th line, 2nd mean series

$$=\{u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)\}\frac{q_1-1}{q_1^2}+\{u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\}\frac{q_2-1}{q_2^2}+, \text{ \&c.}$$

7th line, 3rd mean series

$$=\{u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)\}\frac{(q_1-1)^2}{q_1^3}+\{u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\}\frac{(q_2-1)^2}{q_2^3}+, \text{ \&c.}$$

. . . .

$(2r+1)$th line, rth mean series

$$=\{u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)\}\frac{(q_1-1)^{r-1}}{q_1^r}+\{u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\}\frac{(q_2-1)^{r-1}}{q_2^r}+, \text{ \&c.}$$

$(2r+2)$th line, rth series of remainders

$$\{u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)\}\frac{(q_1-1)^r}{q_1^r}+\{u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\}\frac{(q_2-1)^r}{q_2^r}+, \text{ \&c.}$$

Where when P is put $=0$, 1, 2, &c., successively the expression in each case gives the value of the first, second, third, &c., numbers respectively to the right of the first strong vertical line.

17. We shall now take the sum of all the mean series and put the result by one series only, thus the sum of all the mean series is equal to—

$$u+\{u_1\sin(\mathrm{V}_1+\mathrm{P}b_1)\}\left\{\frac{1}{q_1}+\frac{q_1-1}{q_1^2}+\frac{(q_1-1)^2}{q_1^3}\dots\frac{(q_1-1)^{r-1}}{q_1^r}\right\}$$

$$+\{u_2\sin(\mathrm{V}_2+\mathrm{P}b_2)\}\left\{\frac{1}{q_2}+\frac{q_2-1}{q_2^2}+\frac{(q_2-1)^2}{q_2^3}\dots\frac{(q_2-1)^{r-1}}{q_2^r}\right\}$$

$$+\{u_3\sin(\mathrm{V}_3+\mathrm{P}b_3)\}\left\{\frac{1}{q_3}+\frac{q_3-1}{q_3^2}+\frac{(q_3-1)^2}{q_3^3}\dots\frac{(q_3-1)^{r-1}}{q_3^r}\right\}+,\ \&\text{c.}$$

Suppose $\mathrm{S}=\frac{1}{q}+\frac{q-1}{q^2}+\frac{(q-1)^2}{q^3}+\frac{(q-1)^3}{q^4}\dots\frac{(q_1-1)^{r-1}}{q^r}$,

$$=\frac{1}{q}\left\{1+\left(\frac{q-1}{q}\right)+\left(\frac{q-1}{q}\right)^2+\left(\frac{q-1}{q}\right)^3\dots\left(\frac{q-1}{q}\right)^{r-1}\right\},$$

therefore

$$\mathrm{S}\left(\frac{q-1}{q}\right)=\frac{1}{q}\left\{\left(\frac{q-1}{r}\right)+\left(\frac{q-1}{q}\right)^2+\left(\frac{q-1}{q}\right)^3\dots\left(\frac{q-1}{q}\right)^r\right\}.$$

Hence by subtraction $\mathrm{S}\frac{1}{q}=\frac{1}{q}\left\{1-\left(\frac{q-1}{q}\right)^r\right\}$,

by division $\mathrm{S}=1-\left(\frac{q-1}{q}\right)^r$.

18. From this result we can write the sum of all the mean series—

$$=u+\{u_1\sin(\mathrm{V}_1+\mathrm{P}b_1)\}\left\{1-\left(\frac{q_1-1}{q_1}\right)^r\right\}$$

$$+\{u_2\sin(\mathrm{V}_2+\mathrm{P}b_2)\}\left\{1-\left(\frac{q_2-1}{q_2}\right)^r+,\ \&\text{c.}\right.$$

Thus from one series of observations we get two series, one is the sum of several mean series, and the other the series of last remainders, and of course the sum of these series is equal to the observation series; thus whenever we want the sum of all the mean series we shall get it by subtracting the last remaining series from the corresponding observation series. We shall rewrite these series below and call them *observation series*, *mean series*, and *remaining series* respectively.

Observation series

$$=u+\{u_1\sin(\mathrm{V}_1+\mathrm{P}b_1)\}+\{u_2\sin(\mathrm{V}_2+\mathrm{P}b_2)\}+,\ \&\text{c.}\quad\dots\quad(6).$$

Mean series

$$=u+\{u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)\}\left\{1-\left(\frac{q_1-1}{q_1}\right)^r\right\}$$

$$+\{u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\}\left\{1-\left(\frac{q_2-1}{q_2}\right)^r\right\}+, \text{\&c.} \quad . \quad (7).$$

Remaining series

$$=\{u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)\}\left(\frac{q_1-1}{q_1}\right)^r+\{u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\}\left(\frac{q_2-1}{q_2}\right)^r+, \text{\&c.} \ (8).$$

Throughout these three expressions a particular value of P will correspond to a particular vertical column, the next greater value of P to the next vertical column, and so on.

19. We have up to this point derived these last two results as general with respect to the factors of the type $\frac{1}{q}$ which depend on the value of m, and of this we have only said that it is an odd whole number. Our next step is to see what conclusions we shall arrive at when we apply some particular value to m.

(1.) Suppose that mc is less than K_1, therefore it is also less than each of K_2, K_3, &c., and mb_1, mb_2, &c., are each less than 2π; thus the factors $\frac{\sin\frac{mb_1}{2}}{m\sin\frac{b_1}{2}}$, $\frac{\sin\frac{mb_2}{2}}{m\sin\frac{b_2}{2}}$, &c., or their equivalents $\frac{1}{q_1}$, $\frac{1}{q_2}$, &c., are positive. We have shown before that each of the above factors is a proper fraction, and consequently each of their equivalents is a proper fraction; thus each of q_1, q_2, &c., is greater than *one*, and therefore the value of each of the factors $\left(\frac{q_1-1}{q_1}\right)^r$, $\left(\frac{q_2-1}{q_2}\right)^r$, &c., when r is sufficiently great reduces to zero.

(2.) Suppose that mc is equal to K_1, therefore it is less than each of K_2, K_3, &c., then, as before, we can show that $\frac{1}{q_1}$ is zero, and each of q_2, q_3, &c., is positive, and the value of $\left(\frac{q_1-1}{q_1}\right)^r$ or $\left(1-\frac{1}{q_1}\right)^r$ is *one*, and the value of each of the factors $\left(\frac{q_2-1}{q_2}\right)^r$, $\left(\frac{q_3-1}{q_3}\right)^r$, &c., when r is sufficiently great, reduces to zero. Thus, when mc is equal to the period K_1, and the number of operations of taking successive means and remainders are sufficiently performed, the remaining series will be reduced to $u_1 \sin(\mathrm{V}_1+\mathrm{P}b_1)$, or a subordinate series in the observation whose period is K_1.

(3.) Suppose that mc is greater than K_1 and less than $2\mathrm{K}_1$, and less than each of K_2, K_3, &c., then, as before, we can show that q_1 is negative, and each of q_2, q_2, &c., is positive.

Let $q_1 = -Q_1$, then the factor $\left(\frac{q_1-1}{q_1}\right)^r = \left(1-\frac{1}{q_1}\right)^r$ will be equal to $\left(1+\frac{1}{Q_1}\right)^r = \left(\frac{Q_1+1}{Q_1}\right)^r$, and the value of each of the factors $\left(\frac{q_2-1}{q_2}\right)^r$, $\left(\frac{q_3-1}{q_3}\right)^r$, &c., where r is sufficiently great, reduces to zero. Thus, when mc is greater than the period K_1, less than K_2, and less than double the period of K_1, and the number of operations of taking successive means and remainders are sufficiently performed, the remaining series reduces to $\{u_1 \sin(V_1 + Pb_1)\}\left(\frac{Q_1+1}{Q_1}\right)^r$, or a subordinate series in the observation series whose period is K_1, and the value of each term of the series is greater than the corresponding term of the original series: since $\left(\frac{Q_1+1}{Q_1}\right)$ is greater than *one*, and when r is large enough, $\left(\frac{Q_1+1}{Q_1}\right)^r$ can be made as large as we please.

20. Up to this point we have treated the above subject by supposing that m is an odd number. By simple inspection it will be seen that when m is an even number the results will be quite similar to the one when m is an odd number, but the labour of operations of taking successive means will be twofold. Because the mean of m values from which the mean is taken will not correspond to any of the values in the lot, but will correspond to the middle of some two consecutive values, and for this reason the mean should be written on the line instead of in the column on our form; and after the operations of these first means are performed, we must take the mean of two consecutive means found out first, at both sides of any particular column, and put it in the column. Then these second means thus found are to be subtracted from the corresponding values from which the two sets of means are derived. Similarly from these remainders new means and new remainders are to be found precisely in the same manner as spoken of above. In order to keep our language the same whether m is an odd or an even number, we shall always, in the case where m is an even number, give the name of first set of means, second set of means, &c., to those means found by two of the operations described above.

21. This suggests that we must always prefer m as an odd number, except when there is real need of preferring m as an even number, since the labour of operations, when m is an even number, is greater than when m is an odd number. It will be seen further on that there is sometimes real need of taking m as an even number.

22. The three results which we have found above are very important, and we shall now show their utility, and discuss some more cases from them which we have not yet discussed, by taking some particular examples by which we can show our method of tracing unknown periods.

Suppose we have at our disposal a long series of daily means (say) of barometer observations, and we suspect in it a period of about 25 days. The first thing that we shall do is, that we shall take a small series of daily means and curve it on a curve form, and see generally the period of duration of turning points between successive maxima or between successive minima. Suppose the duration of the period found was from 5 to 10 days, then we shall take the least period of 5 days, that is, we shall adopt 5 as our working value of m, and work the processes as already discussed with the whole series of daily means, until generally each mean of the last mean series is less than 2 (say) in the last place of figures.* We shall take then for further treatment the sum of all the mean series instead of the original series. The sum of all the mean series will be found directly by subtracting the last remaining series from the first daily mean series. We shall take this series for a second set of operations. The number we shall take for operations is $25\left(\frac{180+77{\cdot}5}{180}\right)=$nearly 36, but we shall take 35 because the operations are simpler for odd number than for even number. Now we shall have 35 as our working value of m, by which we shall continue the operations, say up to 5th, and then curve the 5th remaining series on a curve form. If there be only one period in the series from about 17 to 35 days, we shall begin to get simple harmonic waves in the fifth or in some further remaining series; but if there be more than one period the result will be compound, about which we shall discuss presently. But first suppose that there is only one period. If we do not find definitely the duration of simple wave curve in the 5th remaining series, then we shall continue the process on to the 6th remaining series and curve on the same curve form and with the same *zero* line, until the duration of simple wave curve is definitely found. It may here be mentioned, that if the period is not of an exact number of days, we should have a long series to begin with, so that we shall take some exact number of waves from the last remaining series and find the number of days in them, and divide the days by the number of waves, and the result will be the number of days in the period. We must also remark that if there should be no simple harmonic period of exact duration—25 days—that we have been suspecting and searching for, the process we have been performing will still tell us if there be any simple harmonic period of any duration between (say) 20 and 30 days. The only difference we shall have is in these cases that we shall have to increase the number of operations to find the remaining series. In the following table we shall give a few numerical results of the factors—

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* We always have some limit of accuracy of observing the instruments and reducing the observed values; for example, we observe barometer to a nearest thousandth of an inch, and also reduce daily means to a nearest thousandth of an inch. By less than two in the last place of figures, in this case, we mean less than two thousandths of an inch of barometer, and so on.

$$\left\{1-\frac{\sin\frac{35\pi}{20}}{35\sin\frac{\pi}{20}}\right\}^r, \quad \left\{1-\frac{\sin\frac{35\pi}{25}}{35\sin\frac{\pi}{25}}\right\}^r, \text{ and } \left\{1-\frac{\sin\frac{35\pi}{30}}{35\sin\frac{\pi}{30}}\right\}^r$$

	Numerical values of the factors.		
	$\left\{1-\frac{\sin\frac{35\pi}{20}}{35\sin\frac{\pi}{20}}\right\}^r$	$\left\{1-\frac{\sin\frac{35\pi}{25}}{35\sin\frac{\pi}{25}}\right\}^r$	$\left\{1-\frac{\sin\frac{35\pi}{30}}{35\sin\frac{\pi}{30}}\right\}^r$
When $r=1$	1·1291	1·2168	1·1367
„ $r=7$	2·339	3·886	2·452
„ $r=11$......	3·802	8·441	4·094
„ $r=12$......	4·293	10·25	4·654
„ $r=16$......	6·977	22·26	7·770
„ $r=18$......	8·895	32·81	10·04
„ $r=24$......	18·43	105·0	21·66
„ $r=36$......	79·12	1076·0	100·7

The object of introducing the above table is only to give a rough idea to the reader of the rapidity with which the numerical values of these and such similar factors increase after a certain number of operations of the remaining series.

23. Suppose we have suspected only the period of 25 days, but in reality there is not any period between 5 and 35 days; the result will be, as investigated before, that the range of remaining series will be lessened and lessened, until each term of the last remaining series will be less than two in the last place of the figures.

24. Suppose in the daily mean series we have a real existence of the two periods (say) 25 and 20 days, we shall still find the existence of the period of 25 days. Because the numerical value (regardless of sign) of the factor $\frac{\sin\frac{35\pi}{25}}{35\sin\frac{\pi}{25}}$ is greater than the numerical value

(regardless of sign) of the factor $\sin\dfrac{\frac{35\pi}{20}}{35\sin\frac{\pi}{20}}$ (refer to the curve of factors) and consequently $\left(\dfrac{Q_2+1}{Q_2}\right)^r$ can be made as large as we please in comparison with $\left(\dfrac{Q_1+1}{Q_1}\right)^r$ when r is large enough, where $\left(-\dfrac{1}{Q_2}\right)$ stands for the former factor and $\left(-\dfrac{1}{Q_1}\right)$ stands for the latter factor. Thus if the compound result in the last remaining series be curved out on a curve form the maxima and minima of 25 days' period will be apparent. Similarly if any number of compound periods be mixed in the last remaining series, still by sufficiently increasing the number of operations the 25 days' period will be apparent. In conclusion we can say that among the periods which may have mixed in the last remaining series the one of which the factor of the type $\left(\dfrac{1}{q}\right)$ has the greatest numerical value (regardless of sign) will be apparent, whether the period is of 25 days or not.

25. The only case in which the period will not be rendered distinctly apparent is, that when the two periods are such that their factors of the first mean series are equal to one another, and then the two variations will continue combined in the same proportions as in the original observations.

26. We have in the above example stated that when we find systematic movements in the daily mean curve, of which the duration is from 5 to 10 days, we must first work the series with five day operations, &c. It is not, however, essential, when the period which we are seeking is of a considerable range compared with that of the shorter period, then the period can be found without it; but when the range is very small the first process of smoothing is quite essential. It is not also essential that we should stop the operations when the limit of the mean series is reduced to less than 2 in the last place of figures in the series, but we may stop at any stage of the process after considering what reduction will take place in the range of the period we are seeking. This is merely a process for making the series of smooth-flowing numbers without sensibly reducing the range of the larger periods.

27. The process of smoothing series generally in practice is to take the mean of two consecutive numbers, and put it down below and on the line between the two numbers, and so on with each pair of numbers in succession. Next from this new series the same thing is repeated up to a certain number of operations. One effect of this process is to reduce the range of each simple harmonic variation of

which the original series is composed. The extent of this reduction of range is shown in the following expression for the series that is found after S repetitions of such smoothing operations :—

Smoothed observation series*

$$u = +\{u_1 \sin(\mathrm{U}_1 + \mathrm{P}b_1)\} \cos^s \frac{b_1}{2} + \{u_2 \sin(\mathrm{U}_2 + \mathrm{P}b_2)\} \cos^s \frac{b_2}{2} +, \&c.$$

It will be seen at once that as the period is smaller and smaller, we have greater and greater reduction and *vice versâ.*

28. We shall, however, give a clue when a series contains enormous irregularities, to the distribution of each number amongst the adjacent columns, that is effected by our two methods of smoothing. We shall give this by supposing that we have a series whose terms are 13, of which each of the first and last 6 has the value *zero*, and the 7th has the value 64. The result of six smoothing operations of the second method gives 1, 6, 15, 20, 15, 6, 1. From this we see that the reduction for the middle term which occupies the same column as the number 64 in the original set is rapid, and the distribution amongst the adjacent terms is smooth-flowing and of a convergent character.

29. We shall now apply our first method to a series whose terms are 37, of which each of the first and last 18 has the value *zero*, and the 19th has the value 343. Suppose we have a number 7 for a working value of m to start the operations, we shall after three operations get a series of 19 terms whose values are +1, +3, +6, −11, −27, −42, +91, +75, +57, +37, +57, +75, +91, −42, −27, −11, +6, +3, +1. From this we see also that the reduction is rapid for the middle term, and the distribution amongst the adjacent terms is in smooth-flowing waves which grow smaller and smaller and ultimately vanish.

Method of finding the Amplitude of a Variation of known Period.

30. The processes which we are going to apply for this purpose are

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* This may be easily understood if we put $n=2$ in the factor $\dfrac{\sin\frac{nb}{2}}{n\sin\frac{b}{2}}$ of the series 4 in paragraph 7. When $n=2$ the factor becomes $\dfrac{\sin b}{2\sin\frac{b}{2}} = \dfrac{2\sin\frac{b}{2}\cos\frac{b}{2}}{2\sin\frac{b}{2}} = \cos\frac{b}{2}$ for the first operation, and in the same way it becomes $\cos\frac{b}{2}\cos\frac{b}{2}$, *i.e.*, $\cos^2\frac{b}{2}$ for the second operation, thus for S operations it becomes $\cos^s\frac{b}{2}$. This may be taken as a type of the factor for each expression of series of the type $\{u\sin(\mathrm{U}+\mathrm{P}b)\}$.

slightly different from the preceding ones: we shall, therefore, first of all apply them to the general form and then to particular examples.

31. Suppose that $mc = \mathrm{K}_n$ where K_n is a known period and m is some integral number. Then using m as working value let us find as before the first mean series from the observation series, the first series of remainders, and the second mean series; but instead of subtracting the terms of the second mean series from the corresponding terms of the first series of remainders to get the second series of remainders, let us add them in the same order and call this series the first *difference and sum series.* Again from this series let us find, in the same manner as stated above, the second, the third, &c., and the rth *difference and sum series.* The rth *difference and sum series* will, as will presently appear, be of the following form—

$$u_1 \sin(\mathrm{V}_1 + \mathrm{P}b_1)\left(1-\frac{1}{q_1^2}\right)^r + u_2 \sin(\mathrm{V}_2+\mathrm{P}b_2)\left(1-\frac{1}{q_2^2}\right)^r + \ldots$$

$$u_n \sin(\mathrm{V}_n+\mathrm{P}b_n)\left(1-\frac{1}{q_n^2}\right)^r + u_{n+1}\sin(\mathrm{V}_{n+1}+\mathrm{P}b_{n+1})\left(1-\frac{1}{q_{n+1}^2}\right)^r +, \text{ \&c.} \quad . \ . \ . \ . \quad (9).$$

32. The following calculations are made for ready reference and to show how the factors of the form $\left(1-\frac{1}{q^2}\right)^r$, which occurred in our rth *difference and sum series*, are obtained.

The factor for 1st means $=\frac{1}{q}$.

The factor for 1st remainders $=\left(1-\frac{1}{q}\right)$.

The factor for 2nd means $=\left(1-\frac{1}{q}\right)\frac{1}{q}$.

The factor for 1st difference and mean series

$$=\left(1-\frac{1}{q}\right)+\left(1-\frac{1}{q}\right)\frac{1}{q}=\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)=\left(1-\frac{1}{q^2}\right).$$

The factor for 3rd means $=\left(1-\frac{1}{q^2}\right)\frac{1}{q}$.

The factor for 3rd remainders

$$=\left(1-\frac{1}{q^2}\right)-\left(1-\frac{1}{q^2}\right)\frac{1}{q}=\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right).$$

The factor for 4th means $=\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right)\frac{1}{q}$.

The factor for 2nd difference and sum series

$$=\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right)+\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right)\frac{1}{q}=\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)$$

$$=\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q^2}\right)=\left(1-\frac{1}{q^2}\right)^2.$$

Thus it can be shown that the factor for rth *difference and sum series* will be $\left(1-\frac{1}{q^2}\right)^r$.

33. Now the last factors of all the terms in the series (9) contain a term of the type $\frac{1}{q^2}$, so that all these factors are positive, and, as before, it is easy to see that each of them has a tendency to vanish as r increases, except in the cases where the terms of the type $\frac{1}{q^2}$ have the value *zero*. The only terms of the type $\frac{1}{q^2}$, whose values will be *zero*, are those which are formed from the sines of the angles mb_n, $2mb_n$, $3mb_n$, &c., because these angles alone are some integral multiple of 3π. Thus when r is sufficiently great the series (9) will be reduced to—

$$u_n \sin(u_n+\mathrm{P}b_n)+u_{n2}\sin(\mathrm{U}_{n2}+2\mathrm{P}b_n)+u_{n3}\sin(\mathrm{U}_{n3}+3\mathrm{P}b_n)+, \text{ \&c.}$$

If we curved now the first m values in order out of the last horizontal line of our form which forms our rth *difference and sum series*, it will be a variation with its full amplitude of the known period m, and if the next m values be curved in the same way we shall have a repetition of the same curve, and so on.

34. Although we have shown that the factors of the form $\left(1-\frac{1}{q^2}\right)^r$ when $\frac{1}{q^2}$ is not *zero*, and r is sufficiently great, reduce to zero, yet the reduction of the factors which are formed for the periods less than mc is so much slower than the factors which are formed for the periods greater than mc, that it is almost impracticable to continue the process to the limit. We shall therefore in what follows, investigate a particular case of a known period, and we shall find it advantageous to use partly our method and partly the usual method of combining the series with the known period.

35. Suppose we have a series of daily means of barometer observations in which we have detected a period (say) of exactly 36 days, by the method already described. We shall, first of all, in the following table, give factors of the kind in series (9) for different values of r and different values of the periods of the type K.

$\frac{nb}{2}$	90°	108°	257°·4	442°·2	624°·6	805°·8	986°·4	1167°·0	1347°·6
K (Period in days).	72	60	25·2	14·65	10·37	8·04	6·57	5·55	4·81
$\frac{1}{q}$	·63622	·50409	− ·21610	+ ·12640	− ·08850	+ ·06721	− ·05348	+ ·04368	− ·03626
$\frac{1}{q^2}$	·40478	·25411	·04670	·01597	·00784	·00452	·00286	·00191	·00131
$1-\frac{1}{q^2}$	·59522	·74789	·95330	·98403	·99216	·99548	·99714	·99809	·99869
$\left(1-\frac{1}{q^2}\right)^5$	·07471	·2309	·78732	·92260	·96140	·97758	·98583	·99049	·99346
$\left(1-\frac{1}{q^2}\right)^{10}$	·00558	·05330	·61987	·85134	·92428	·95567	·97186	·98107	·98697
$\left(1-\frac{1}{q^2}\right)^{15}$	·00042	·01230	·48804	·78551	·88859	·93423	·95808	·97168	·98051

In the above table, the periods less than *mc* of the type K which we have selected are those that have maxima or minima values of their first factors of the type $\frac{1}{q}$ so that their factors of the type $\left(\frac{q^2-1}{q^2}\right)^r$ will have the greatest possible reduction, that is, factors for intermediate periods will reduce at slower rates.

36. We also see that when r is equal to 5, we have the factor corresponding to the period 72 days reduced to $\frac{1}{13}$, and from this we can infer that the factors corresponding to greater periods than 72 days will be reduced to less than $\frac{1}{13}$. So that for all practical purposes we shall make the operations 5, and assume practically that in the fifth *difference and sum series* we have reduced all series of the periods 72 and greater than 72 days to *zero*, and retained in their full magnitude all the series of the periods less than 36 days, whilst the series of periods intermediate between 36 and 72 days will be only partially obliterated. Such a series may be represented by

$$\{u_1 \sin(\mathrm{U}_1+\mathrm{P}b_1)+u_2 \sin(\mathrm{U}_2+\mathrm{P}b_2)+, \&c.\}$$

$$+\left\{u_n \sin\left(\mathrm{U}+\mathrm{P}\frac{2\pi}{36}\right)+u_{n2} \sin\left(\mathrm{U}_{n2}+\mathrm{P}\frac{4\pi}{36}\right)+, \&c.\right\}$$

$$+\left\{u_{n+1} \sin(\mathrm{U}_{n+1}+\mathrm{P}b_{n+1})+u_{n+2} \sin(\mathrm{U}_{n+2}+\mathrm{P}b_{n+2})\left(\frac{q^2_{n+2}-1}{q^2_{n+2}}\right)^5+, \&c.\right.$$

In the first pair of brackets of the above expression, we have arranged those terms of the simple harmonic series whose periods are less than the period of 36 days, but not any of the period of the form $\frac{36}{\mathrm{N}}$ where N is any integral number; in the second pair of brackets of the expression, we have arranged those terms of the simple harmonic series whose periods are of the form $\frac{36}{N}$, that is, it contains all the harmonics of 36 days; and in the third pair of brackets of the expression, we have arranged those terms of the simple harmonic series whose periods are greater than the period 36 days. Suppose in the fifth *difference and sum series*, or in the last horizontal line of our form, we have 36M values where M is some large integral number. Suppose we have another table ruled with more than M horizontal and 36 vertical lines. In the first horizontal line let us enter 36 numerical values from the commencement of the series in order, in the second horizontal line the next 36 numerical values, and so on, in the Mth horizontal line the last 36 numerical values. Let the sum and mean of each vertical column be taken in the $(m + 1)$th and $(m + 2)$th horizontal lines respectively; then these means are the subject of our present investigation.

37. The general term of the simple harmonic series in the first horizontal line is—

$$u \sin (U+Pb);$$

the general term of the same simple harmonic series in the same vertical column of the second horizontal line is—

$$u \sin (U+Pb+36b);$$

the general term of the same simple harmonic series in the same vertical column of the third horizontal line is—

$$u \sin \{U+Pb+2\ (36b)\};$$

the general term of the same simple harmonic series in the same vertical column of the Mth horizontal line is—

$$u \sin \{U+Pb+(M-1)(36b)\};$$

thus the mean of this vertical column will be—

$$u \sin \left\{ (U+Pb)+\frac{M-1}{2}36b \right\} \frac{\sin M\frac{36b}{2}}{M \sin \frac{36b}{2}},$$

then the general series may be written—

$$u \sin \left\{ \left(U+\frac{M-1}{2}36b \right)+Pb \right\} \sin \frac{M\frac{36b}{2}}{M \sin \frac{36b}{2}}.$$

Thus the last horizontal line of means in our new form, which is composed of 36 values, will be of the form—

$$\left[u_1 \sin\left\{\left(\mathrm{U}_1+\frac{\mathrm{M}-1}{2}\,36b_1\right)+\mathrm{P}b_1\right\}\frac{\sin \mathrm{M}\,\frac{36b_1}{2}}{\mathrm{M}\sin\frac{36b_1}{2}}\right.$$

$$\left.+u_2 \sin\left\{\left(u_2+\frac{\mathrm{M}-1}{2}36\frac{b_2}{2}\right)+\mathrm{P}b_2\right\}\frac{\sin \mathrm{M}\,\frac{36b_2}{2}}{\mathrm{M}\sin\frac{36b_2}{2}}\right]$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$+\left\{u_n \sin\left(\mathrm{U}_n+\mathrm{P}\frac{2\pi}{36}\right)+u_{n2}\sin\left(\mathrm{U}_{n2}+\mathrm{P}\frac{4\pi}{36}\right)+,\ \&\mathrm{c.}\right\}$$

$$+\left[u_{n+1}\sin\left\{\left(\mathrm{U}_{n+1}+\frac{\mathrm{M}-1}{2}36b_{n+1}\right)+\mathrm{P}b_{n+1}\right\}\frac{\sin \mathrm{M}\,\frac{36b_{n+1}}{2}}{\mathrm{M}\sin\frac{36b_{n+1}}{2}}\times\left(\frac{q^2_{n+1}-1}{q^2_{n+1}}\right)^r\right.$$

$$\left.+,\ \&\mathrm{c.}\right].$$

And it can be shown that the last factors of the terms in the first and the third pairs of brackets, where M is sufficiently great, reduce to zero (see curve of factors in fig. 1). Thus the 36 mean values found in the last horizontal line of our new form will represent the variation of the 36-day period.

38. Next suppose that the period which we have detected is not an exact number of days but (say) of 36·4 days, we shall still take the nearest integral number, viz., 36, as our working value of *m*, and by it find (say) the 5th *difference and sum series* as already described. In this case the amplitude of the variation of the period of 36·4 days will be slightly reduced in the 5th *difference and sum series.* The reduction of the 1st simple harmonic variation that is of the period of 36·4 days will be (1—·99954), that of 18·2 days will be (1—·99954), that of 12·13 will be (1—·99954), and so on. The next thing is to show how the series is to be combined, which we are going to show in the following manner.

39. In the first horizontal line of our new form, we shall enter first 36 values. In the second horizontal line 8 values, from the beginning of the next 37 values, will be entered up to the 8th vertical column, but in the 9th vertical column we shall enter the mean of 9th and 10th values out of the 37 values intended for the second horizontal line; in the rest of the horizontal line the remaining values will be entered. The object of combining the 9th and 10th values and entering the mean result in the 9th vertical column of the second horizontal line is that, since we are dealing with exact days only, we must enter the values in the columns in such a manner, that the error

of distribution must not be greater than half a day, and so the mean of the 9th and 10th values, which was found more appropriate to 9th column than any other column, was entered in the 9th column. Similarly up to the *m*th horizontal line, the values will be entered by properly combining two values for entering in one column as many times as need be in the course of entering the series in the manner stated above. Also the sums and means of each vertical column are taken in the same way as in the preceding example. We will not go into details, as in the preceding example, but can simply say that the last numbers of our new form will be the variation of the period of 36·4 days with its almost full magnitude, and for 36 points, the intervals between which from point to point will be $1\frac{1}{90}$ days.

40. We should remark that the process of *difference and sum series* has also an obvious advantage in separating the observation series into any number of difference and sum series without affecting the series by the process. By affecting the series by the process, we mean that in the former process for the *remaining series* and *mean series*, if we take any arbitrary number as our working value of m, and if there should be periodicities lying between $\frac{mc}{2}$ and mc, we shall get the periodicities mixed up in both the series with a greater amplitude than the original series. These separations are very useful for the first trials when suspecting approximate periods.

Application of the Process and Detection of a Period of about $9\frac{1}{2}$ *Months.*

41. For this purpose we have taken means of barometric pressure at Bombay, from 1847 to 1872 and 1873 to 1878, thus forming our Table I. The first part of monthly means is also given in Table I, page 11, of the volume entitled "The Meteorology of the Bombay Presidency," by Charles Chambers, Esq., F.R.S., Superintendent of the Colaba Observatory, Bombay; and the last part at the foot of Tables I to VI, pages 58 to 63 of the "Bombay Magnetical and Meteorological Observations, 1871 to 1878." The description of the barometer and mode of observation, &c., are fully given in both the volumes. With the numbers in Table I, we have performed one operation to get the first *difference and sum series* by taking 12 as our working value of m, and the result is entered in Table II. The numbers in Table II are subtracted from the corresponding numbers in Table 1 and entered in Table III. At the foot of Table II we have entered the mean for January in the 24 years, 1848 to 1871, the mean for February in the 24 years, the mean for March, &c., &c., up to December.* These means form the annual variation of barometric

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* Properly speaking, we should have taken the means of thirty years, 1848 to 1877, instead of 1848 to 1871; but first of all we have taken for our discussion the

pressure for the 24 years. If we add the constant 29·808, the annual mean of 26 years, 1847 to 1872, to the 12 numbers in our Table II, then these numbers will be found almost identical with the numbers at the foot of Table I, given in the Meteorology of the Bombay Presidency. The 12 numbers at the foot of Table II were then subtracted in order from each of the rows of 12 numbers commencing from January, and ending with December for each of the years 1848 to 1877 of the same table, and the remainders entered in Table IV. Again, with the numbers in Table IV, we have performed three operations, taking 3 as our working value of m to get the third *difference and sum series,* which we have given in Table V, then the numbers in Table V were subtracted from the corresponding numbers in Table IV, and entered in Table VI.

We need not say that the sum of the numbers in Tables III, V, and VI, from July, 1848, to June, 1876, and successive repetitions of the corresponding numbers of annual variation at the foot of Table II, taken in order, will be equal to the corresponding numbers in Table I. The numbers in Tables III and VI are curved in figs. 2 and 3 respectively. We presume that the curve of fig. 2 is composed of periods of more than 12 months, and that of fig. 3 is of less than 12 months and greater than three months. By looking at the oscillations of the curve of fig. 2 we suspect that amongst the periods there are periods of 11 or 12 years, 4½ years, and 2 years. By looking at the oscillations of the curve of fig. 3, we see that amongst the periods there is a great predominance of a 9½-monthly period. This we have ascertained in the following manner:—We have first of all marked by a cross (thus ×) all the minima that have their time-interval from one minimum to next minimum 8 to 11 months, or any integral multiple of 8 to 11 months, and then filled up the rest 8 to 11-monthly minima that were less marked by a mark (thus 0). It is remarkable that the oscillations are more prominent and of greater amplitude from the year 1849 to 1862 than from the year 1863 to 1876. If we count the marks of minima commencing from July, 1848, and ending with November, 1876, they will be found to be 37; thus there are 36 periods in 340 months, *i.e.*, 9½ months, as an average duration for the period.

42. Again, we find that there are intermediate minima which are of less amplitude than already stated, and the duration of which is also from 8 to 11 months. These also we have marked by marks (* and 0), as in the preceding example. In the present case both these

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series from 1847 to 1872, and have made all the operations we are going to describe. But afterwards it was thought proper to add to the above result the six years 1873 to 1878, which were available. But as the work was far advanced it was not worth while to go into the trouble of doing the work over again for the sake of including those six years.

marks are made on the upper side of the curve, while in the preceding case on the lower side of it, to distinguish them from each other.

43. We find from May, 1858, to October, 1866, that the occurrences of these minima have disappeared. From December, 1848, to May, 1858, there are 12 periods, that is $\frac{113}{12}=9\frac{1}{2}$ months for a period, and from October, 1866, to April, 1877, there are 13 periods, *i.e.*, $\frac{126}{13}=9\frac{3}{4}$ months for a period. From this we might guess that to form such a curve there must be two prominent periods, which may be as (say) 5 and $9\frac{1}{2}$ months.

44. To clear this doubt, we have further taken the numbers in Table VI, and performed five operations to get the fifth *difference and sum series* by taking 5 as our working value of m. These numbers we have given in Table VII and curved in fig. 4. Then the numbers in Table VII were subtracted from the corresponding numbers in Table VI and entered in Table VIII, and curved in dotted lines with the corresponding curve of fig. 3, and with the same scale and on the same zero line. Now, by looking at the oscillations of the dotted curve, it appears that it is of a simple harmonic character, and it confirms the $9\frac{1}{2}$-monthly period as already described. But in both these curves (continuous and dotted) there are a few exceptions to the confirmation of the $9\frac{1}{2}$-monthly period. These exceptional periods are—from March, 1857, to December, 1859; June, 1863, to September, 1866; April, 1868, to January, 1871; and November, 1872, to the end.

45. We shall now turn to fig. 4. By looking at the oscillations of this curve we again see that amongst the periods there is a great predominance of a $4\frac{3}{4}$-monthly period. This we have ascertained as in the case of $9\frac{1}{2}$-monthly period. If we count all the marks made on the curve, we have 65; thus there are 64 periods in 302 months (from June, 1850, to August, 1875), and $\frac{302}{64}=4{\cdot}72$, *i.e.*, nearly $4\frac{3}{4}$ months,* which is an average duration for the period.

We should now here remark that, although in the continuous curve of fig. 3 we found that the occurrences of the subordinate minima have disappeared from May, 1858, to October, 1866, still they appear in the $4\frac{3}{4}$-monthly period curve of fig. 4.

Application of the Method to find the Duration of an exact Period.

46. The writer had taken a small series of daily means of barometer observations, derived from hourly tabulations of barograph, uncorrected to standard barometer No. 58, from 1st December, 1875, to 31st December, 1876, for trial, when the method of detecting any

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* The writer found worth noticing that the period of $4\frac{3}{4}$ months is almost an exact multiple of the $36\frac{1}{2}$-day period which he has found further on in paragraph 50.

simple harmonic waves of an approximately known period was in its infancy. The aim was to see whether on barometer observations there was any influence of lunar period, the duration of which is about 30 days. The trial which we are going to describe was made in the earlier period of 1878, when the barograph tabulation was under consideration. Daily means of barograph tabulations, corrected to standard barometer No. 58, for this same period, may be found in the "Bombay Magnetical and Meteorological Observations, 1871 to 1878," pp. 60 and 61. The difference between the daily means, corrected and uncorrected to standard instrument, is very small, and not varying from day to day more than (say) about two or three thousandths of an inch. The treatment which was given to these daily means and the results which followed from it may be considered satisfactory.

47. In order to get the series in smooth-flowing numbers, we have taken 15 as our working value of m, and found first mean series. Then this series has been taken for further operation, and by taking 45 as our working value of m, we have found first, second, &c., up to fifth series of remainders. The first, third, and fifth series of remainders are curved in fig. 5 in plain, dotted, and interrupted lines respectively, with the same scale and on the same zero line. Looking at the plain curve, we see that there are nine turning-points of minima, the duration thereof being moderately uniform. If we take the first minimum in the middle of the small gap that occurs in the curve, we have in 298 days eight periods. Thus an average duration of one period is about 37 days. Again, if we count the days from the second minimum, we have 258 days in seven waves, and thus an average duration is still about 37 days. Thus, instead of the lunar period of 30 days, we find a 37-day period. From this it is clear that if there be lunar period in the series at all, the amplitude of it must be much smaller than the 37-day period.

48. The object of drawing the three curves on the same zero line and with the same scale is to show graphically the process of swelling the waves as the operations increase and of their appearance, as in the case of the third wave, which disappears in the first series of remainders. The object is also to impress the necessity of making smooth-flowing numbers before taking them for operation; as in the case of second, third, fifth, and seventh waves, all the irregular movements in the first series of remainders are found to appear in the third and fifth series of remainders. With these remarks we conclude without giving the numbers from which the present curves are formed.

Second Application.

49. In order to test the reality of the period of about 37 days on a large scale, we have taken a series of weekly means of barometer observations at Bombay, commencing from 1st January, 1848, and

ending with the 2nd January, 1852, as the year 1848 was the first complete year when the system of hourly observations began without interruption at the Colaba Observatory. The weekly means were chosen in place of daily means for two reasons, first that as no observations were made on Sundays and a few holidays, no continuous daily means were available during the period; second, to save comparatively the labour of calculations of the processes. On some occasions when there were atmospheric disturbances, or for similar reasons, observations were made on Sundays and none on Mondays. For this and similar other reasons the weekly means were calculated from all the daily means from Sunday to Saturday.

These means are given in Table IX, and curved in fig. 6 in an interrupted line. With these weekly means we have performed 5 operations to get the fifth series of remainders. The numbers in the first and fifth series of remainders are also given in Tables X and XI respectively, and curved with the interrupted line of fig. 6, and with the same scale in dotted and continuous lines respectively.

50. Looking at the continuous curve, it will be seen that from 24th week of 1848 to 8th week of 1849, we have 7 waves of simple harmonic character, and equidistant in duration. Thus there are 36 weeks, or 36 × 7 days in 7 periods, so that the average duration of a period is exactly 36 days. Again, from 8th week of 1849 to 24th week of 1850 there are 13 waves, also almost equidistant in duration. Thus there are 68 weeks, or 68 × 7 days in 13 periods, so that the average duration of a period is nearly 36·6 days. Again, from 24th week of 1850 to 34th week of 1851 there are, we presume, 12 periods. It will be observed that out of these there are 7 periods, viz., 1st and 7th to 12th, whose waves are quite prominent; with regard to the remaining 5 periods their waves seem to be almost obscured by irregularities. Thus there are 63 weeks, or 63 × 7 days, in 12 periods, so that the average duration of a period is nearly 36·7 days. If we now add all the weeks and periods together, to get the mean duration of the period we have 167 weeks and 32 periods, therefore the average duration of the period is nearly 36·53 days.

51. We should here remark that as we are dealing with weekly values, there is a possibility of an error, either too much or too little, of one week in the whole period of 167 weeks. If we take 166 weeks for 32 periods, we get an average duration of 36·31 days; and if we take 168 weeks, we get 36·75 days. Thus the period we have found out is approximate to the extent above mentioned. We shall, therefore, say that the period is approximately $36\frac{1}{2}$ days instead of 36·53 days.

52. The object of drawing three curves of weekly means on the same zero line, and with the same scale, is the same as in the preceding example of the curves of daily means. The formation into regular

waves of five weekly period of the continuous curve from the numbers corresponding to the interrupted curve for the periods from 31st to 37th week of 1848, and 13th to 23rd week of 1849 is worth noticing. It should also be noticed that the irregular movements that exist in the interrupted curve for the period from 31st week of 1850 to 15th week of 1851, also exist in the continuous line curve.

53. Let us now combine the result by taking $36\frac{1}{2}$ days as an average duration of the period to get an average range or amplitude of the period, as shown in paragraph 39.

Table showing the variations of $36\frac{1}{2}$-day period for several groups of periods and for the whole period.

Number of points in the curve (7·3 days for a point).	0.	1.	2.	3.	4.
First group of 7 periods	− ·113	− ·016	+ ·114	+ ·081	− ·058
Second group of 13 periods ..	− ·041	− ·034	+ ·018	+ ·034	+ ·016
Third group of 12 periods ...	− ·018	+ ·016	+ ·028	+ ·004	− ·019
Combined above three groups of 32 periods...........	− ·048	− ·011	+ ·041	+ ·033	− ·013
Combined whole series.......	− ·048	− ·012	+ ·036	+ ·035	− ·013

These results are curved in figures 7 to 11 respectively.

It must be mentioned here that the ranges which we see in these results are not exact, but enlarged by our process and

$$\left\{1-\frac{\sin 7\frac{7\pi}{36\frac{1}{2}}}{\sin 7\frac{7\pi}{36\frac{1}{2}}}\right\}^5$$

= nearly 2·7, is the extent to which our results are enlarged.

But we must also notice that the facts that the period is not an integral number of weeks, that our method of combination is in consequence rather rough, and that the number of points in the curve is too small, tend to reduce the range of the period. If the process had been applied to the continuous daily means instead of to weekly means the results would have been of a larger and equal amplitude of several groups of the period.

54. The writer feels himself under great obligation to his superior, C. Chambers, Esq., F.R.S., Superintendent of the Bombay Observatory, for the warm interest he took in carefully reading this tract, and in giving here and there some very valuable suggestions.

Table I.

Year.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
1847...	29·917	29·903	29·872	29·791	29·722	29·621	29·651	29·707	29·748	29·825	29·893	29·897
1848...	·927	·916	·830	·770	·741	·649	·654	·706	·812	·826	·909	·922
1849...	·945	906	·870	·779	·714	·624	·672	·700	·727	·841	·889	·911
1850...	·878	·927	·885	·816	·763	·639	·643	·736	·786	·785	·891	·949
1851...	·958	·901	·857	·779	·739	·631	·599	·697	·782	·828	·838	·946
1852...	·929	·903	·829	·794	·746	·630	·660	·731	·756	·856	·905	·923
1853...	·940	·873	·859	·797	·775	·644	·673	·742	·793	·854	·863	·955
1854...	·935	·921	·885	·792	·760	·674	·618	·712	·715	·785	·923	·933
1855...	·935	·932	·853	·818	·787	·663	·658	·743	·787	·826	·939	·944
1856...	·985	·922	·835	·787	·718	·651	·610	·707	·799	·834	·897	·932
1857...	·930	·869	·831	·790	·735	·640	·649	·681	·811	·857	·918	·981
1858...	·921	·939	·852	·798	·674	·698	·670	·720	·778	·822	·927	·944
1859...	·933	·918	·878	·787	·789	·662	·628	·731	·779	·854	·851	·931
1860...	·953	·875	·845	·827	·764	·650	·631	·725	·735	·809	·917	·919
1861...	·907	·897	·857	·764	·713	·666	·639	·689	·774	·857	·867	·932
1862...	·920	·871	·862	·801	·786	·634	·625	·689	·693	·780	·853	·887
1863...	·939	·883	·838	·746	·734	·606	·637	·697	·767	·811	·895	·947
1864...	·945	·923	·887	·821	·815	·674	·660	·742	·794	·867	·920	·931
1865...	·972	·910	·874	·800	·741	·669	·656	·666	·788	·848	·890	·918
1866...	·959	·905	·867	·816	·775	·675	·661	·706	·795	·808	·927	·969
1867...	·965	·916	·879	·819	·734	·671	·667	·692	·750	·826	·983	·981
1868...	·937	·921	·881	·831	·796	·673	·699	·742	·817	·857	·920	·963
1869...	·966	·936	·879	·821	·773	·659	·668	·705	·730	·824	·908	·904
1870...	·879	·877	·836	·795	·764	·651	·637	·724	·754	·807	·900	·936
1871...	·894	·891	·863	·793	·768	·641	·672	·733	·779	·821	·870	·932
1872...	·932	·907	·850	·776	·767	·640	·658	·677	·760	·820	·866	·892
1873...	·924	·895	·859	·803	·733	·646	·641	·742	·788	·824	·935	·943
1874...	·975	·925	·857	·826	·720	·628	·655	·720	·740	·801	·920	·953
1875...	·918	·905	·849	·780	·772	·652	·655	·720	·754	·833	·923	·938
1876...	·922	·904	·868	·758	·763	·683	·631	·728	·802	·875	·896	·957
1877...	·976	·936	·887	·827	·790	·723	·745	·762	·808	·865	·921	·917
1878...	·953	·953	·902	·820	·783	·683	·654	·665	·688	·780	·836	·882

Table II.

Year.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
1848...	+ ·130	+ ·118	+ ·031	− ·031	− ·061	− ·154	− ·150	− ·099	+ ·006	+ ·020	+ ·103	+ ·116
1849...	+ ·141	+ ·102	+ ·068	− ·022	− ·086	− ·175	− ·125	− ·099	− ·072	+ ·042	+ ·088	+ ·110
1850...	+ ·076	+ ·124	+ ·080	+ ·010	− ·044	− ·167	− ·164	− ·071	− ·020	− ·020	+ ·086	+ ·144
1851...	+ ·155	+ ·100	+ ·058	− ·020	− ·058	− ·165	− ·196	− ·098	− ·013	+ ·033	+ ·042	+ ·150
1852...	+ ·133	+ ·106	+ ·030	− ·006	− ·055	− ·173	− ·144	− ·074	− ·050	+ ·049	+ ·097	+ ·114
1853...	+ ·131	+ ·063	+ ·048	− ·016	− ·038	− ·170	− ·142	− ·074	− ·023	+ ·039	+ ·049	+ ·141
1854...	+ ·122	+ ·110	+ ·075	− ·017	− ·048	− ·133	− ·187	− ·092	− ·090	− ·021	+ ·115	+ ·124
1855...	+ ·125	+ ·120	+ ·038	+ ·002	− ·031	− ·156	− ·162	− ·077	− ·033	+ ·007	+ ·120	+ ·128
1856...	+ ·170	+ ·110	+ ·025	− ·022	− ·089	− ·153	− ·194	− ·095	− ·002	+ ·034	+ ·097	+ ·133
1857...	+ ·131	+ ·069	+ ·030	− ·013	− ·070	− ·166	− ·158	− ·128	+ ·001	+ ·046	+ ·106	+ ·168
1858 .	+ ·107	+ ·125	+ ·038	− ·016	− ·140	− ·116	− ·144	− ·095	− ·036	+ ·009	+ ·112	+ ·129
1859	+ ·118	+ ·104	+ ·064	− ·027	− ·025	− ·151	− ·184	− ·080	− ·031	+ ·046	+ ·043	+ ·125
1860...	+ ·147	+ ·070	+ ·040	+ ·024	− ·039	− ·152	− ·170	− ·075	− ·064	+ ·010	+ ·119	+ ·122
1861...	+ ·111	+ ·101	+ ·061	− ·032	− ·084	− ·130	− ·158	− ·109	− ·025	+ ·059	+ ·069	+ ·135
1862...	+ ·124	+ ·076	+ ·068	+ ·009	− ·004	− ·154	− ·161	− ·095	− ·089	+ ·001	+ ·073	+ ·107
1863...	+ ·159	+ ·102	+ ·057	− ·037	− ·052	− ·183	− ·155	− ·098	− ·033	+ ·006	+ ·086	+ ·135
1864...	+ ·129	+ ·104	+ ·065	− ·003	− ·012	− ·154	− ·169	− ·087	− ·034	+ ·039	+ ·094	+ ·108
1865...	+ ·150	+ ·090	+ ·057	− ·016	− ·073	− ·144	− ·156	− ·146	− ·024	+ ·037	+ ·079	+ ·105
1866...	+ ·145	+ ·091	+ ·051	− ·001	− ·043	− ·145	− ·159	− ·114	− ·025	− ·012	+ ·106	+ ·148
1867...	+ ·144	+ ·096	+ ·058	− ·002	− ·086	− ·150	− ·155	− ·130	− ·074	+ ·000	+ ·156	+ ·152
1868...	+ ·107	+ ·089	+ ·047	− ·004	− ·040	− ·164	− ·138	− ·096	− ·021	+ ·021	+ ·086	+ ·131
1869 ..	+ ·136	+ ·109	+ ·055	·000	− ·044	− ·155	− ·142	− ·102	− ·074	+ ·022	+ ·108	+ ·107
1870...	+ ·083	+ ·081	+ ·040	·000	− ·031	− ·145	− ·160	− ·074	− ·044	+ ·008	+ ·099	+ ·133
1871...	+ ·092	+ ·087	+ ·059	− ·012	− ·037	− ·164	− ·134	− ·073	− ·027	+ ·016	+ ·065	+ ·127
1872...	+ ·128	+ ·104	+ ·049	− ·024	− ·032	− ·158	− ·139	− ·119	− ·035	+ ·024	+ ·071	+ ·096
1873...	+ ·127	+ ·096	+ ·058	·000	− ·072	− ·161	− ·168	− ·069	− ·026	+ ·009	+ ·120	+ ·129
1874...	+ ·161	+ ·111	+ ·044	+ ·015	− ·089	− ·180	− ·152	− ·086	− ·065	− ·004	+ ·115	+ ·148
1875...	+ ·113	+ ·100	+ ·043	− ·026	− ·034	− ·155	− ·152	− ·088	− ·054	+ ·024	+ ·115	+ ·129
1876...	+ ·112	+ ·093	+ ·056	− ·055	− ·052	− ·135	− ·189	− ·094	− ·023	+ ·047	+ ·065	+ ·123
1877...	+ ·139	+ ·097	+ ·046	− ·016	− ·055	− ·123	− ·100	− ·083	− ·035	+ ·024	+ ·083	+ ·083
Mean...	+ ·128	+ ·098	+ ·052	− ·010	− ·054	− ·155	− ·159	− ·095	− ·037	+ ·020	+ ·092	+ ·129

Table III.

Year.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
1848...	29·797	29·798	29·799	29·801	29·802	29·803	29·804	29·805	29·806	29 806	29·806	29·806
1849...	·804	·804	·802	·801	800	·799	·797	·799	·799	·799	·801	·801
1850...	·802	·803	·805	·806	·807	·806	·807	·807	·806	·805	·805	·805
1851...	·803	·801	·799	·799	·797	·796	·795	·795	·795	·795	·796	·796
1852...	·796	·797	·799	·800	·801	·803	·804	·805	·806	·807	·808	·809
1853...	·809	·810	·811	·813	·813	·814	·815	·816	·816	·815	·814	·814
1854...	·813	·811	·810	·809	·808	·807	·805	·804	·805	·806	·808	·809
1855...	·810	·812	·815	·816	·818	·819	·820	·820	·820	·819	·819	·816
1856...	·815	·812	·810	·809	·807	·804	·804	·802	·801	·800	·800	·799
1857...	·799	·800	·801	·803	·805	·806	·807	·809	·810	·811	·812	·813
1858...	·814	·814	·814	·814	·814	·814	·814	·815	·814	·813	·815	·815
1859...	·815	·814	·814	·814	·814	·813	·812	·811	·810	·808	·808	·806
1860...	·806	·805	·805	·803	·803	·802	·801	·800	·799	·799	·798	·797
1861..	·796	·796	·796	·796	·797	·796	·797	·798	·799	·798	·798	·797
1862...	·796	·795	·794	·792	·790	·788	·786	·784	·782	·781	·780	·780
1863...	·780	·781	·781	·783	·786	·789	·792	·795	·800	·805	·809	·812
1864...	·816	·819	·822	·824	·827	·828	·829	·831	·828	·828	·826	·823
1865...	·822	820	·817	·816	·814	·813	·812	·812	·812	·811	·811	·813
1866...	·814	·814	·816	·817	·818	·820	·820	·820	·820	·820	·821	·821
1867...	·821	·820	·821	·821	·820	·821	·822	·822	·824	·825	·827	·829
1868...	·830	·832	·834	·835	·836	·837	·837	·838	838	·836	·834	·832
1869.	·830	·827	·824	·821	·817	·814	·810	·807	·804	·802	·800	·797
1870...	·796	·796	·796	·795	·795	·796	·797	·798	·798	·799	·801	·803
1871...	·802	·804	·804	·805	·805	·805	·806	·806	·806	·805	·805	·805
1872...	·804	·803	·801	·800	·799	·798	·797	·796	·795	·796	·795	·796
1873...	·797	·799	·801	·803	·805	·807	·809	·811	·814	·815	·815	·814
1874...	·814	·814	·813	·811	·809	·808	·807	·806	·805	·805	·805	·805
1875...	·805	·805	·806	·806	·806	·807	·807	·808	·808	·809	·808	·809
1876...	·810	·811	·812	·813	·815	·818	·820	·822	·825	·828	·831	·834
1877...	·837	·839	·841	·843	·845	·846	·845	·845	·843	·841	·838	·834

Table IV.

Year.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
1848...	+ ·002	+ ·020	− ·021	− ·021	− ·007	+ ·001	+ ·009	− ·004	+ ·043	·000	+ ·011	− ·013
1849...	+ ·013	+ ·004	+ ·016	− ·012	− ·032	− ·020	+ ·034	− ·004	− ·035	− ·022	− ·004	− 019
1850...	− ·052	+ ·026	+ ·028	+ ·020	+ ·010	− ·012	− ·005	+ ·024	+ ·017	− ·040	− ·006	+ ·015
1851...	+ ·027	+ ·002	+ ·006	− ·010	− ·004	− ·010	− ·037	− ·003	+ ·024	+ ·013	− ·050	+ ·021
1852...	+ ·005	+ ·008	− ·022	+ ·004	− ·001	− ·018	+ ·015	+ ·021	− ·013	+ ·029	+ ·005	− ·015
1853...	+ ·003	− ·035	− ·004	− ·006	+ ·016	− ·015	+ ·017	+ ·021	+ ·014	+ ·019	− ·043	+ ·012
1854...	− ·006	+ ·012	+ ·023	− ·007	+ ·006	+ ·022	− ·028	+ ·003	− ·057	− ·041	+ ·023	− ·005
1855...	− ·003	+ ·022	− ·014	+ ·012	+ ·023	− ·001	− ·003	+ ·018	+ ·004	− ·013	+ ·028	− ·001
1856...	+ ·042	+ ·012	− ·027	− ·012	− ·035	+ ·002	− ·035	·000	+ ·035	+ ·014	+ ·005	+ ·004
1857...	+ ·003	− ·029	− ·022	− ·003	− ·016	− ·011	+ ·001	− ·033	+ ·038	+ ·026	+ ·014	+ ·039
1858...	− ·021	+ ·027	− ·014	− ·006	− ·086	+ ·039	+ ·015	·000	+ ·001	− ·011	+ ·020	·000
1859...	− ·010	+ ·006	+ ·012	− ·017	+ ·029	+ ·004	− ·025	+ ·015	+ ·006	+ ·026	− ·049	− ·004
1860...	+ ·019	− ·028	− ·012	+ ·034	+ ·015	+ ·003	− ·011	− ·020	− ·027	− ·010	+ ·027	− ·007
1861...	− ·017	+ ·003	+ ·009	− ·022	− ·030	+ ·020	+ ·001	− ·014	+ ·012	+ ·039	− ·023	+ ·006
1862...	− ·004	− ·022	+ ·016	+ ·019	+ ·050	+ ·001	− ·002	·000	− ·052	− ·019	− ·019	− ·022
1863...	+ ·031	+ ·004	+ ·005	− ·027	+ ·002	− ·028	+ ·004	− ·003	+ ·004	− ·014	− ·006	+ ·006
1864...	+ ·001	+ ·006	+ ·013	+ ·007	+ ·042	+ ·001	− ·010	+ ·008	+ ·003	+ ·019	+ ·002	− ·021
1865...	+ ·022	− ·008	+ ·005	− ·006	− ·019	+ ·011	+ ·003	− ·051	+ ·013	+ ·017	− ·013	− ·024
1866...	+ ·017	− ·007	− ·001	+ ·009	+ ·011	+ ·010	·000	− ·019	+ ·012	− ·032	+ ·014	+ ·019
1867...	+ ·016	− ·002	+ ·006	+ ·008	− ·032	+ ·005	+ ·004	− ·035	− ·037	− ·019	+ ·064	+ ·023
1868...	− ·021	− ·009	− ·005	+ ·006	+ ·014	− ·009	+ ·021	− ·001	+ ·016	+ ·001	− ·006	+ ·002
1869...	+ ·008	+ ·011	+ ·003	+ ·010	+ ·010	·000	+ ·017	− ·007	− ·037	+ ·002	+ ·016	− ·022
1870...	− ·045	− ·017	− ·012	+ ·010	+ ·023	+ ·010	− ·001	+ ·021	− ·007	− ·012	+ ·007	+ ·004
1871...	− ·036	− ·011	+ ·007	− ·002	+ ·017	− ·009	+ ·025	+ ·022	+ ·010	− ·004	− ·027	− ·002
1872...	·000	+ ·006	− ·003	− ·014	+ ·022	− ·033	+ ·020	− ·024	− ·002	+ ·004	− ·021	− ·033
1873...	− ·001	− ·002	+ ·006	+ ·010	− ·018	− ·006	− ·009	+ ·026	+ ·011	− ·011	+ ·028	·000
1874...	+ ·033	+ ·013	− ·008	+ ·025	− ·035	− ·025	+ ·007	+ ·009	− ·028	− ·024	+ ·023	+ ·019
1875...	− ·015	+ ·002	− ·009	− ·016	+ ·020	·000	+ ·007	+ ·007	− ·017	+ ·004	+ ·023	·000
1876...	− ·016	− ·005	+ ·004	− ·045	+ ·002	+ ·020	− ·030	+ ·001	+ ·014	+ ·027	− ·027	− ·006
1877...	+ ·011	− ·001	− ·006	− ·006	− ·001	+ ·032	+ ·059	+ ·012	+ ·002	+ ·004	− ·009	− ·046

Table V.

Year.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
1848...	...	...	...	...	...	...	+ ·001	− ·014	+ ·021	− ·010	+ ·004	− ·008
1849...	+ ·006	− ·005	+ ·010	+ ·003	− ·013	− ·011	+ ·030	− ·005	− ·028	+ ·021	+ ·007	·000
1850...	− ·027	+ ·018	+ ·013	− ·004	+ ·004	− ·008	− ·011	+ ·014	+ ·015	− ·025	+ ·001	+ ·008
1851...	+ ·006	− ·009	+ ·004	− ·006	+ ·005	+ ·009	− ·017	− ·004	+ ·019	+ ·009	− ·036	+ ·022
1852...	+ ·003	+ ·001	− ·015	+ ·010	+ ·006	− ·017	+ ·011	+ ·013	− ·014	− ·006	+ ·007	− ·005
1853...	+ ·016	− ·015	+ ·003	− ·003	+ ·012	− ·014	+ ·001	+ ·003	·000	+ ·014	− ·028	+ ·019
1854...	− ·007	− ·001	+ ·016	− ·015	·000	+ ·017	− ·021	+ ·025	− ·019	− ·015	+ ·029	− ·012
1855...	− ·009	+ ·018	− ·018	+ ·001	+ ·013	− ·008	− ·009	+ ·014	− ·002	− ·012	+ ·014	− ·020
1856...	+ ·015	+ ·003	− ·015	+ ·008	− ·009	+ ·021	− ·017	− ·006	+ ·018	− ·007	− ·006	+ ·002
1857...	+ ·009	− ·011	− ·002	+ ·011	− ·007	+ ·003	+ ·011	− ·028	+ ·022	− ·001	− ·010	+ ·019
1858...	− ·026	+ ·015	+ ·001	+ ·014	− ·044	+ ·036	+ ·002	− ·011	+ ·003	− ·009	+ ·014	− ·001
1859...	− ·011	+ ·006	+ ·008	− ·023	+ ·020	+ ·005	− ·017	+ ·002	+ ·002	+ ·023	− ·035	+ ·005
1860...	+ ·028	− ·022	− ·010	+ ·019	− ·004	− ·005	+ ·002	+ ·004	− ·009	− ·005	+ ·021	− ·006
1861...	− ·014	+ ·006	+ ·015	− ·009	− ·016	+ ·024	− ·001	− ·019	+ ·005	+ ·024	− ·027	+ ·009
1862...	+ ·007	− ·014	+ ·006	− ·004	+ ·018	− ·016	− ·001	+ ·018	− ·019	+ ·009	+ ·005	− ·014
1863...	+ ·019	− ·004	+ ·001	− ·010	+ ·015	− ·014	+ ·006	+ ·001	+ ·005	− ·007	·000	+ ·007
1864...	− ·003	+ ·001	·000	− ·011	+ ·019	− ·006	− ·009	+ ·008	− ·006	+ ·010	·000	− ·016
1865...	+ ·019	− ·009	+ ·004	+ ·001	− ·016	+ ·017	+ ·012	− ·034	+ ·015	+ ·017	− ·010	− ·013
1866...	+ ·019	− ·006	− ·005	+ ·003	− ·001	+ ·004	− ·002	− ·009	+ ·025	− ·017	− ·007	+ ·010
1867...	+ ·005	− ·012	+ ·004	+ ·013	− ·024	+ ·007	+ ·014	− ·004	− ·008	− ·020	+ ·034	+ ·001
1868...	− ·021	+ ·006	+ ·002	+ ·002	+ ·006	− ·012	+ ·011	− ·008	+ ·006	·000	− ·005	·000
1869...	+ ·003	+ ·003	+ ·005	+ ·003	·000	− ·009	+ ·014	+ ·004	− ·022	+ ·006	+ ·019	− ·007
1870...	− ·013	+ ·011	− ·003	− ·002	+ ·009	− ·005	− ·009	+ ·016	− ·008	− ·012	+ ·010	+ ·011
1871...	− ·019	+ ·002	+ ·009	− ·005	+ ·009	− ·015	+ ·006	+ ·003	+ ·002	+ ·003	− ·012	+ ·006
1872...	·000	+ ·005	− ·001	− ·010	+ ·012	− ·010	+ ·014	− ·017	+ ·002	+ ·014	− ·005	− ·012
1873...	+ ·011	− ·003	− ·001	+ ·009	− ·008	+ ·001	− ·008	+ ·015	·000	− ·018	+ ·014	− ·007
1874...	+ ·012	− ·005	− ·016	+ ·028	− ·016	− ·012	+ ·013	+ ·014	− ·014	− ·014	+ ·018	+ ·006
1875...	− ·017	+ ·009	·000	− ·013	+ ·015	− ·008	·000	+ ·009	− ·014	− ·002	+ ·014	·000
1876...	− ·014	+ ·007	+ ·018	− ·030	+ ·009	+ ·023	− ·022	− ·003	+ ·006	+ ·017	− ·021	− ·001
1877...	+ ·013	− ·002	·000	+ ·001	− ·010	·000						

Table VI.

Year.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
1848...	...	...	...	...	...	...	+ ·008	+ ·010	+ ·002	+ ·010	+ ·007	− ·005
1849...	+ ·007	+ ·009	+ ·006	− ·015	− ·019	− ·009	+ ·004	+ ·001	− ·007	+ ·001	− ·011	− ·019
1850...	− ·025	+ ·008	+ ·015	+ ·024	+ ·006	− ·004	+ ·006	+ ·010	+ ·002	− ·015	− ·007	+ ·007
1851...	+ ·021	+ ·011	+ ·002	− ·004	− ·009	− ·019	− ·020	+ ·001	+ ·005	+ ·004	− ·014	− ·001
1852...	+ ·002	+ ·007	− ·007	− ·006	− ·007	− ·001	+ ·004	+ ·008	+ ·001	+ ·035	− ·002	− ·010
1853...	− ·013	− ·020	− ·007	− ·003	+ ·004	− ·001	+ ·016	+ ·018	+ ·014	+ ·005	− ·015	− ·007
1854...	+ ·001	+ ·013	+ ·007	+ ·008	+ ·006	+ ·005	− ·007	− ·022	− ·038	− ·026	− ·006	+ ·007
1855...	+ ·006	+ ·004	+ ·004	+ ·011	+ ·010	+ ·007	+ ·006	+ ·004	+ ·006	− ·001	+ ·014	+ ·019
1856...	+ ·027	+ ·009	− ·012	− ·020	− ·026	− ·019	− ·018	+ ·006	+ ·017	+ ·021	+ ·011	+ ·002
1857...	− ·006	− ·018	− ·020	− ·014	− ·009	− ·014	− ·010	− ·005	+ ·016	+ ·027	+ ·024	+ ·020
1858...	+ ·005	+ ·012	− ·015	− ·020	− ·042	+ ·003	+ ·013	+ ·011	− ·002	− ·002	+ ·006	+ ·001
1859...	+ ·001	·000	+ ·004	+ ·006	+ ·009	− ·001	− ·008	+ ·013	+ ·004	+ ·003	− ·014	− ·009
1860...	− ·009	− ·006	− ·002	+ ·015	+ ·019	+ ·008	− ·013	− ·024	− ·018	− ·005	+ ·006	− ·001
1861...	− ·003	− ·003	− ·006	− ·013	− ·014	− ·004	+ ·002	+ ·005	+ ·007	+ ·015	+ ·004	− ·003
1862...	− ·011	− ·008	+ ·010	+ ·023	+ ·032	+ ·017	− ·001	− ·018	− ·033	− ·028	− ·024	− ·008
1863...	+ ·012	+ ·008	+ ·004	− ·017	− ·013	− ·014	− ·002	− ·004	− ·001	− ·007	− ·006	− ·001
1864...	+ ·004	+ ·005	+ ·013	+ ·018	+ ·023	+ ·007	− ·001	·000	+ ·009	+ ·009	+ ·002	− ·005
1865...	+ ·003	+ ·001	+ ·001	− ·007	− ·003	− ·006	− ·009	− ·017	− ·002	·000	− ·003	− ·011
1866...	− ·002	− ·001	+ ·004	+ ·006	+ ·012	+ ·006	+ ·002	− ·010	− ·013	− ·015	+ ·021	+ ·009
1867...	+ ·011	+ ·010	+ ·002	− ·005	− ·008	− ·002	− ·010	− ·031	− ·029	+ ·001	+ ·030	+ ·022
1868...	·000	− ·015	− ·007	+ ·004	+ ·008	+ ·003	+ ·010	+ ·007	+ ·010	+ ·001	− ·001	+ ·002
1869...	+ ·005	+ ·008	+ ·008	+ ·007	+ ·010	+ ·009	+ ·003	− ·011	− ·015	− ·004	− ·003	− ·015
1870...	− ·032	− ·028	− ·009	+ ·012	+ ·014	+ ·015	+ ·008	+ ·005	− ·001	·000	− ·003	− ·007
1871...	− ·017	− ·013	− ·002	+ ·003	+ ·008	+ ·006	+ ·019	+ ·019	+ ·008	− ·007	− ·015	− ·008
1872...	·000	+ ·001	− ·002	− ·004	+ ·010	+ ·007	+ ·006	− ·007	− ·004	− ·010	− ·016	− ·021
1873...	− ·012	+ ·001	+ ·007	+ ·001	− ·010	− ·007	− ·001	+ ·011	+ ·011	+ ·007	+ ·014	+ ·007
1874...	+ ·021	+ ·018	+ ·008	− ·003	− ·019	− ·013	− ·006	− ·005	− ·014	− ·010	+ ·005	+ ·013
1875...	+ ·002	− ·007	− ·009	− ·003	+ ·005	+ ·008	+ ·007	− ·002	− ·003	+ ·006	+ ·009	·000
1876...	− ·002	− ·012	− ·014	− ·015	− ·007	− ·003	− ·008	+ ·004	+ ·008	+ ·010	− ·006	− ·005
1877 ..	− ·002	+ ·001	− ·006	− ·005	+ ·018	+ ·030						

Table VII.

Year.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
1850...	...	...	+·009	+·010	−·008	−·013	+·002	+·011	+·006	−·011	−·007	+·001
1851...	+·009	+·001	−·002	·000	+·002	−·006	−·008	+·008	+·007	+·004	−·014	−·002
1852...	+·003	+·009	−·003	−·002	−·001	·000	−·001	−·002	−·008	+·024	−·006	−·006
1853...	−·003	−·004	+·007	+·002	+·002	−·010	+·002	+·005	+·006	+·001	−·012	−·001
1854...	+·002	+·009	−·004	−·004	−·002	+·007	+·005	·000	−·013	−·004	+·006	+·008
1855...	+·001	−·004	−·005	+·001	+·001	+·001	+·002	+·001	·000	−·010	·000	·000
1856...	+·013	+·002	−·006	−·002	−·002	+·003	−·006	+·003	+·002	+·004	·000	−·002
1857...	−·001	−·005	·000	+·004	+·007	−·001	−·005	−·008	+·003	+·005	+·001	−·001
1858...	−·005	+·012	−·002	−·001	−·022	+·011	+·012	+·005	−·007	−·007	+·003	+·002
1859...	·000	−·002	+·001	+·002	+·004	−·002	−·012	+·009	+·001	+·006	−·005	·000
1860...	·000	−·004	−·006	+·005	+·011	+·006	−·005	−·011	−·002	+·002	+·009	·000
1861...	−·003	+·001	+·001	−·001	−·003	+·001	+·001	−·002	−·003	+·006	+·002	+·005
1862...	−·008	−·009	−·001	+·003	+·011	+·001	−·001	−·001	−·006	·000	−·004	·000
1863...	+·011	+·003	+·003	−·010	−·003	−·002	+·006	+·004	+·001	−·002	−·004	+·002
1864...	·000	−·003	·000	+·003	−·008	−·003	−·008	−·003	+·005	+·006	−·001	−·007
1865...	+·003	+·001	+·003	−·004	+·003	+·001	−·001	−·008	+·004	+·004	+·002	−·005
1866...	·000	−·002	·000	−·002	+·003	+·004	+·003	−·003	−·005	−·010	+·015	+·001
1867...	·000	+·001	−·006	−·004	·000	+·012	+·009	−·013	−·018	·000	+·021	+·012
1868...	−·006	−·015	−·004	+·007	+·007	−·002	·000	−·002	+·004	−·002	−·002	·000
1869...	+·002	+·001	−·001	−·002	+·002	+·004	+·001	−·007	−·008	+·006	+·013	+·003
1870...	−·010	−·010	·000	+·010	+·005	+·001	−·005	−·002	−·002	+·003	+·005	−·002
1871...	−·005	−·002	+·004	+·002	·000	−·007	+·003	+·007	+·003	−·005	−·009	+·001
1872...	+·006	+·004	−·003	−·007	+·004	+·002	+·003	−·004	+·002	+·002	·000	−·007
1873...	−·003	+·005	+·008	+·003	−·009	−·005	−·001	+·008	+·003	−·004	−·001	−·008
1874...	+·005	+·006	+·002	−·002	−·010	·000	+·007	+·004	−·007	−·008	+·004	+·010
1875...	+·003	−·009	−·007	−·001	+·005	+·006	+·003	−·006	−·006	+·002		

Table VIII.

Year.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
1850...	...	...	+·006	+·014	+·014	+·009	+·004	−·001	−·004	−·004	·000	+·006
1851...	+·012	+·010	+·004	−·004	−·011	−·013	−·012	−·007	−·002	·000	·000	+·001
1852...	−·001	−·002	−·004	−·004	−·006	−·001	+·005	+·010	+·009	+·011	+·004	−·004
1853...	−·010	−·016	−·014	−·005	+·002	+·009	+·014	+·013	+·008	+·004	−·003	−·006
1854...	−·001	+·004	+·011	+·012	+·008	−·002	−·012	−·022	−·025	−·022	−·012	−·001
1855...	+·005	+·008	+·009	+·010	+·009	+·006	+·004	+·003	+·006	+·009	+·014	+·019
1856...	+·014	+·007	−·006	−·018	−·024	−·022	−·012	+·003	+·015	+·017	+·011	+·004
1857...	−·005	−·013	−·020	−·018	−·016	−·013	−·005	+·003	+·013	+·022	+·023	+·021
1858...	+·010	·000	−·013	−·019	−·020	−·008	+·001	+·006	+·005	+·005	+·003	−·001
1859...	+·001	+·002	+·003	+·004	+·005	+·001	+·004	+·004	+·003	−·003	−·009	−·009
1860...	−·009	−·002	+·004	+·010	+·008	+·002	−·008	−·013	−·016	−·007	−·003	−·001
1861...	·000	−·004	−·007	−·012	−·011	−·005	+·001	+·007	+·010	+·009	+·002	−·002
1862...	−·003	+·001	+·011	+·020	+·021	+·016	·000	−·017	−·027	−·028	−·020	−·008
1863...	+·001	+·005	+·001	−·007	−·010	−·012	−·008	−·008	−·002	−·005	−·002	−·003
1864...	+·004	+·008	+·013	+·015	+·015	+·010	+·007	+·003	+·004	+·003	+·003	+·002
1865...	·000	·000	−·002	−·003	−·006	−·007	−·008	−·009	−·006	−·004	−·005	−·006
1866...	−·002	+·001	+·004	+·008	+·009	+·002	−·001	−·007	−·008	−·005	+·006	+·008
1867...	+·011	+·009	+·008	−·001	−·008	−·014	−·019	−·018	−·011	+·001	+·009	+·010
1868...	+·006	·000	−·003	−·003	+·001	+·005	+·010	+·009	+·006	+·003	+·001	+·002
1869...	+·003	+·007	+·009	+·009	+·008	+·005	+·002	−·004	−·007	−·010	−·016	−·018
1870...	−·022	−·018	−·009	+·002	+·009	+·014	+·013	+·007	+·001	−·003	−·008	−·005
1871...	−·012	−·011	−·006	+·001	+·008	+·013	+·016	+·012	+·005	−·002	−·006	−·009
1872...	−·006	−·003	+·001	+·003	+·006	+·005	+·003	−·003	−·006	−·012	−·016	−·014
1873...	−·009	−·004	−·001	−·002	−·001	−·002	·000	+·003	+·008	+·011	+·015	+·015
1874...	+·016	+·012	+·006	−·001	−·009	−·013	−·013	−·009	−·007	−·002	+·001	+·003
1875...	−·001	+·002	−·002	−·002	·000	+·002	+·004	+·004	+·003	+·004		

Table IX.

Week.	1848.	1849.	1850.	1851.	Week.	1848.	1849.	1850.	1851.
1..	29·931	29·955	29·924	26·965	27..	29·736	29·678	29·720	29·613
2..	·923	·930	·849	·934	28..	·600	·740	·649	·536
3..	·918	·926	·843	·982	29..	·620	·688	·583	·697
4..	·923	·962	·878	·917	30..	·650	·609	·557	·571
5..	·965	·948	·920	·942	31..	·692	·583	·735	·713
6..	·969	·975	·969	·877	32..	·693	·704	·697	·737
7..	·917	·904	·929	·915	33..	·693	·748	·748	·695
8..	·885	·830	·906	·868	34..	·722	·740	·752	·663
9..	·818	·911	·901	·885	35..	·756	·665	·747	·724
10..	·800	·879	·883	847	36..	·778	·695	·778	·714
11..	·865	·937	·879	·811	37..	·834	·715	·742	·846
12..	·838	·853	·884	·882	38..	·855	·642	·811	·836
13..	·826	·781	·902	·864	39..	·781	·832	·801	·753
14..	·803	·788	·852	·800	40..	·712	·838	·816	·762
15..	·846	·765	·795	·810	41..	·851	·781	·806	·834
16..	·721	·755	·829	·731	42..	·882	·859	·755	·817
17..	·704	·795	·801	·683	43..	·853	·889	·752	·937
18..	·777	·780	·792	·730	44..	·825	·846	·817	·802
19..	·759	·722	·789	·795	45..	·871	·817	·846	·829
20..	·754	·704	·758	·761	46..	·881	·915	·818	·804
21..	·706	·703	·764	·704	47..	·962	·911	·942	·902
22..	·693	·668	·728	·645	48..	·981	·925	·961	·926
23..	·651	·662	·667	·533	49..	·901	·848	·928	·975
24..	·571	·625	·592	·703	50..	·899	·885	·933	·929
25..	·618	·565	·646	·652	51..	·926	·950	·981	·966
26..	·750	·639	·650	·587	52..	·953	·976	·962	·930
					53..	..	..	·936	

Table X.

Week.	1848.	1849.	1850.	1851.	Week.	1848.	1849.	1850.	1851.
1..	..	+·019	+·023	+·004	27..	+·087	+·029	+·092	−·010
2..	..	−·013	−·057	+·029	28..	−·067	+·097	·000	−·088
3..	..	−·024	−·065	+·039	29..	−·057	+·025	−·073	+·061
4..	−·012	+·019	−·024	−·023	30..	−·019	−·070	−·113	−·081
5..	+·036	+·023	+·021	+·016	31..	+·025	−·104	+·061	+·054
6..	+·055	+·053	+·062	−·035	32..	+·004	+·027	+·009	+·051
7..	+·020	−·012	+·017	+·022	33..	−·019	+·070	+·032	+·007
8..	−·003	−·082	−·006	−·010	34..	−·016	+·047	+·009	−·064
9..	−·052	+·013	−·006	+·016	35..	−·006	−·036	−·007	−·021
10..	−·050	+·008	−·015	−·020	36..	+·004	−·025	+·010	−·033
11..	+·031	+·083	−·008	−·040	37..	+·057	−·017	−·036	+·089
12..	+·010	+·008	+·013	+·039	38..	+·060	−·096	+·025	+·055
13..	+·012	−·042	+·041	+·043	39..	−·032	+·066	+·014	−·042
14..	+·003	−·023	+·003	+·003	40..	−·112	+·044	+·033	−·064
15..	+·058	−·023	−·041	+·024	41..	+·028	−·031	+·012	+·014
16..	−·056	−·014	+·006	−·042	42..	+·057	+·022	−·044	−·002
17..	−·062	+·037	−·001	−·076	43..	+·014	+·040	−·049	+·111
18..	+·025	+·034	+·002	−·015	44..	−·050	−·014	−·002	−·044
19..	+·028	−·010	+·009	+·074	45..	−·023	−·063	+·004	−·031
20..	+·033	−·015	+·001	+·068	46..	−·015	+·036	−·048	−·078
21..	+·004	+·008	+·037	+·008	47..	+·059	+·033	+·050	+·021
22..	+·014	+·004	+·022	−·040	48..	+·064	+·032	+·045	+·022
23..	−·027	+·010	−·019	−·122	49..	−·028	−·068	−·004	+·056
24..	−·104	−·024	−·089	+·069	50..	−·041	−·032	−·016	
25..	−·042	−·089	−·019	+·042	51..	−·009	+·042	+·029	
26..	+·101	−·018	+·006	−·030	52..	+·026	+·080	+·006	
					53..	..	..	−·027	

Vinayek Narayen Nene.

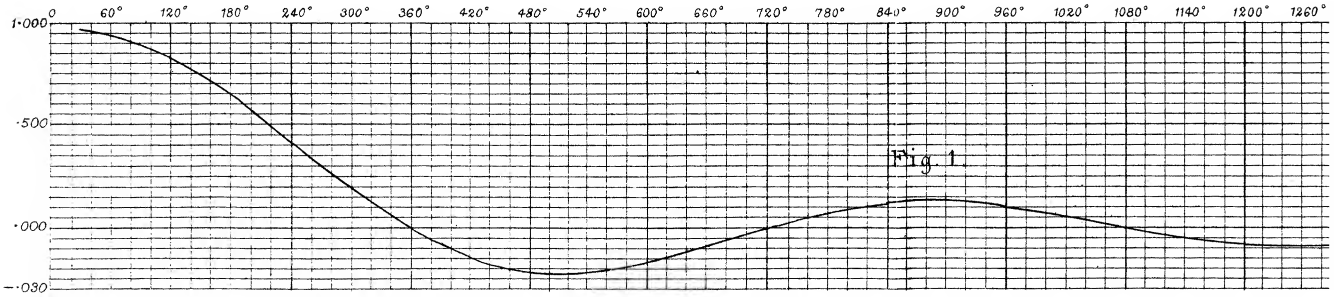


Fig. 1.

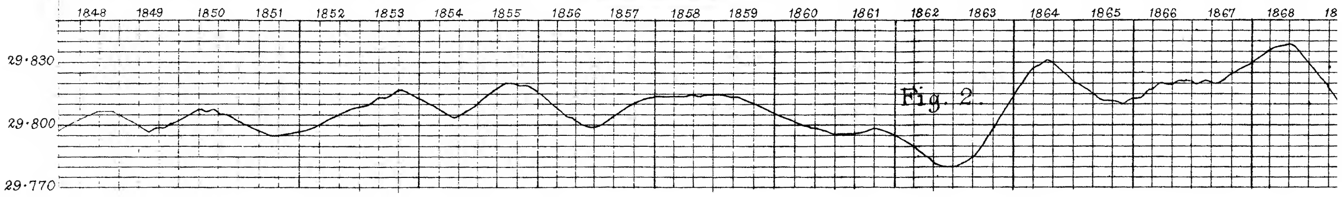


Fig. 2.

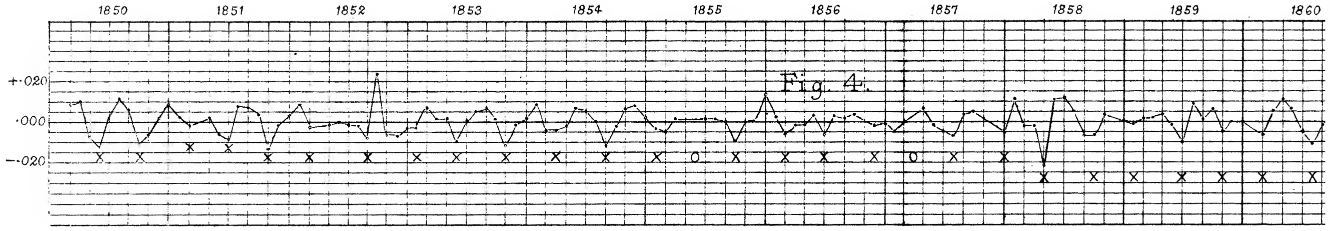


Fig. 4.

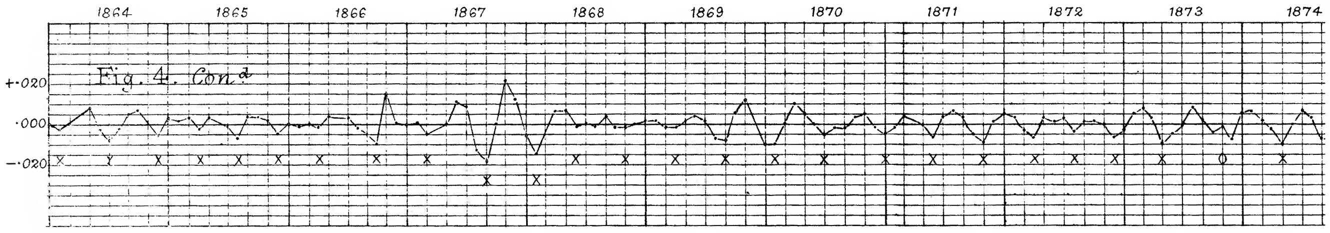
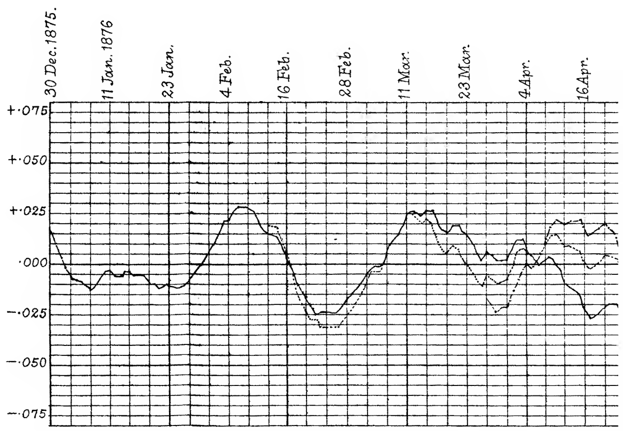
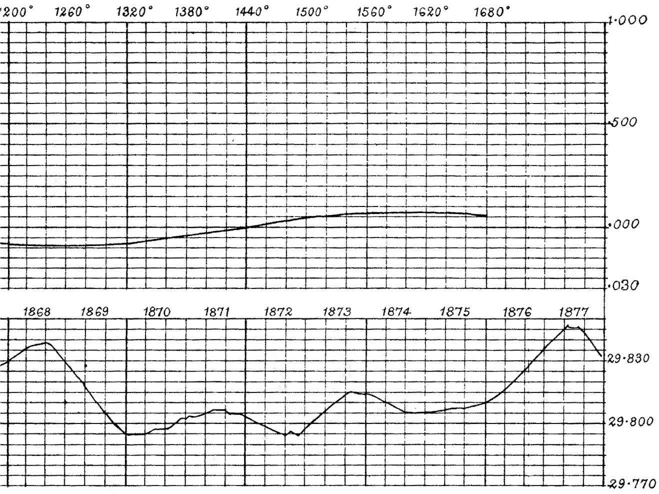


Fig. 4. Con^d





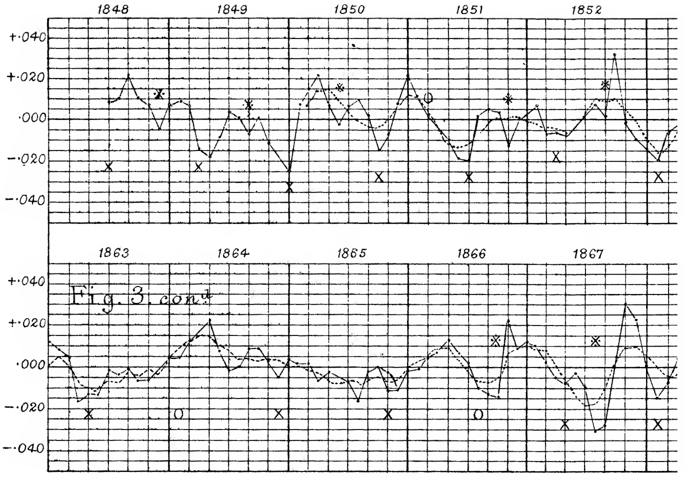
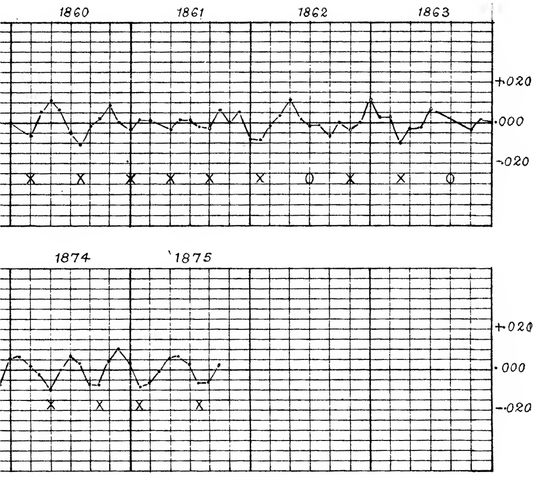


Fig. 3. cond.



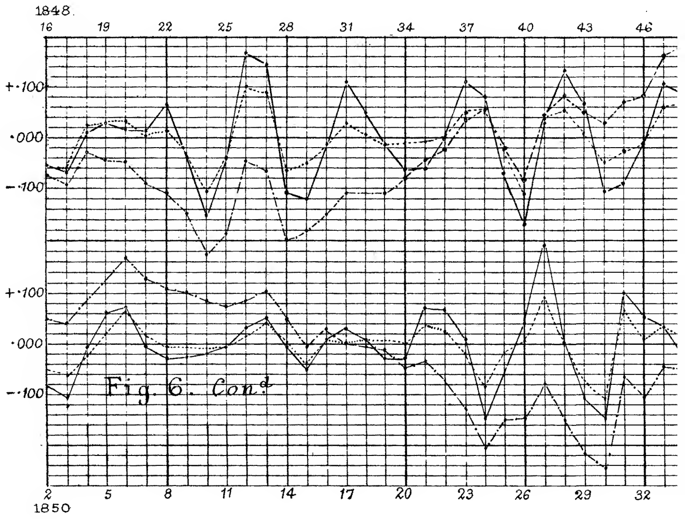


Fig. 6. Cond.

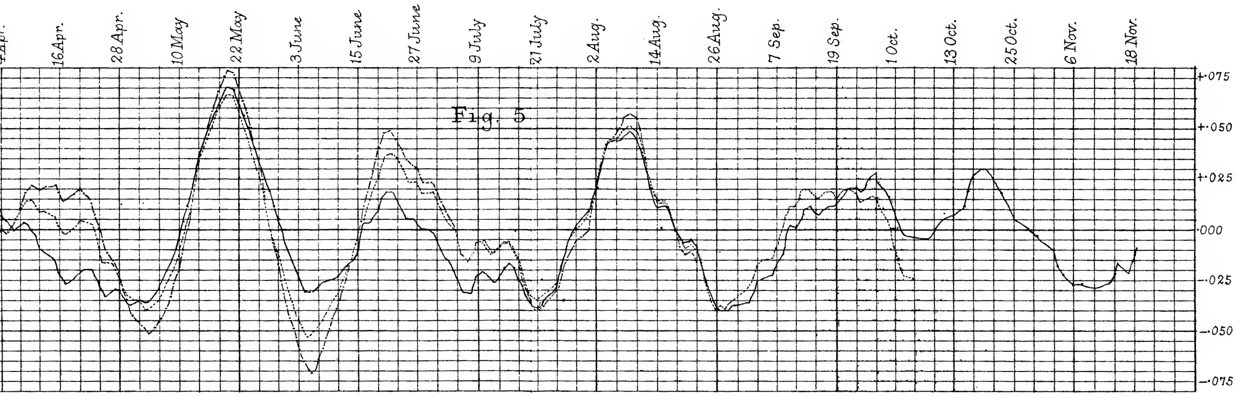
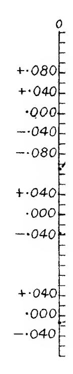


Fig. 5



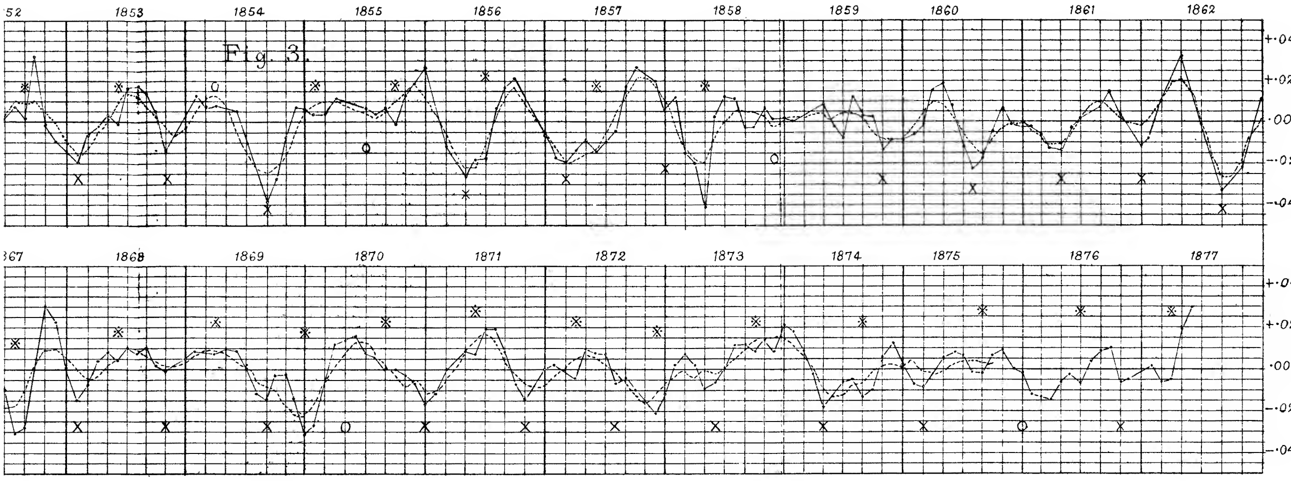


Fig. 3.

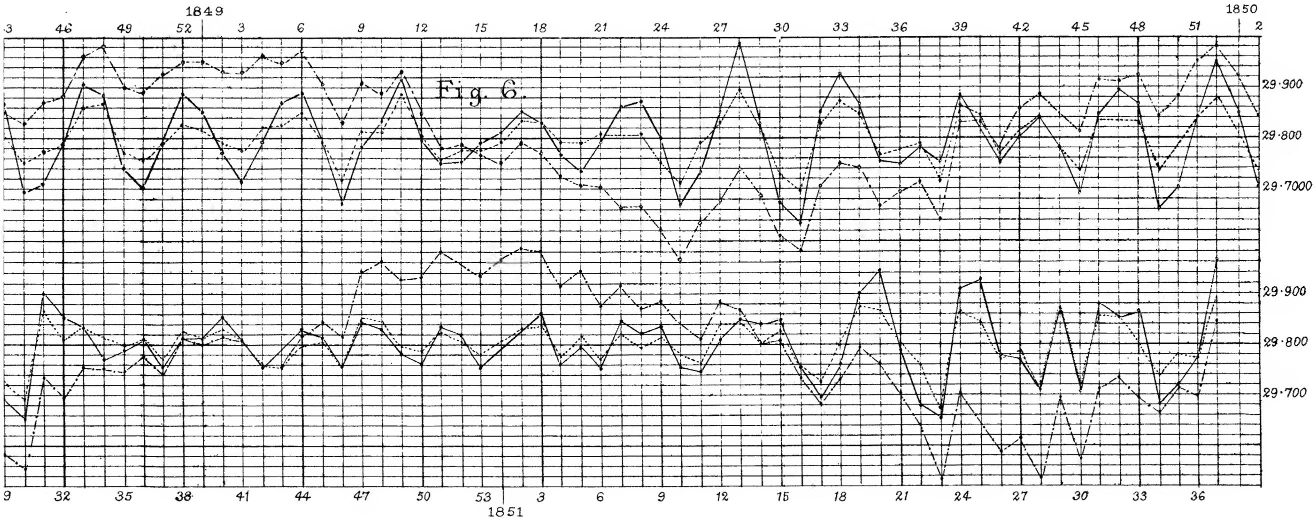


Fig. 6.

36½ day period.

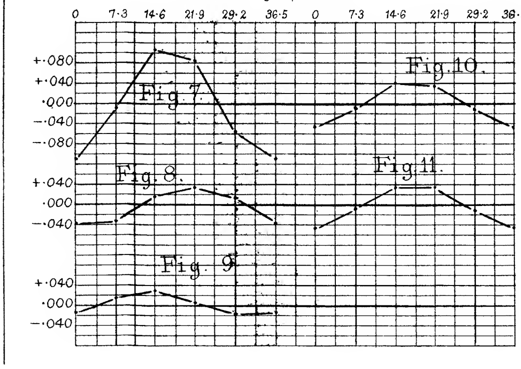


Fig. 7.

Fig. 8.

Fig. 9.

Fig. 10.

Fig. 11.

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Table XI.

Week.	1848.	1849.	1850.	1851.	Week.	1848.	1849.	1850.	1851.
1..	..	+·052	+·049	−·014	27..	+·143	+·051	+·191	−·024
2..	..	−·035	−·084	+·023	28..	−·108	+·185	−·003	−·092
3..	..	−·087	−·106	+·060	29..	−·120	+·036	−·105	+·072
4..	..	−·009	−·012	−·040	30..	−·016	−·129	−·151	−·097
5..	..	+·067	+·061	−·004	31..	+·106	−·171	+·101	+·080
6..	..	+·085	+·071	−·055	32..	+·053	+·052	+·058	+·051
7..	..	−·015	−·003	+·045	33..	−·013	+·130	+·034	+·062
8..	..	−·134	−·032	+·021	34..	−·060	+·064	−·035	−·120
9..	..	−·020	−·024	+·037	35..	−·056	−·047	−·017	−·073
10..	..	+·033	−·020	−·047	36..	−·006	−·056	+·008	−·031
11..	..	+·118	−·009	−·058	37..	+·113	−·020	−·044	+·162
12..	..	−·004	+·030	+·004	38..	+·080	−·046	+·008	
13..	..	−·051	+·047	+·046	39..	−·074	+·088	+·012	
14..	..	−·047	−·006	+·039	40..	−·175	+·027	+·054	
15..	..	−·012	−·057	+·046	41..	+·044	−·049	+·004	
16..	−·055	+·011	+·004	−·050	42..	+·137	+·006	−·049	
17..	−·073	+·055	+·029	−·112	43..	+·067	+·043	−·020	
18..	+·011	+·033	+·005	−·042	44..	−·103	−·018	+·025	
19..	+·031	−·037	−·034	+·101	45..	−·094	−·112	+·017	
20..	+·019	−·064	−·034	+·142	46..	−·016	+·048	−·052	
21..	+·018	−·011	+·064	−·015	47..	+·112	+·098	+·043	
22..	+·062	+·061	+·063	−·120	48..	+·084	+·065	+·027	
23..	−·033	+·073	+·006	−·143	49..	−·060	−·140	−·020	
24..	−·154	−·001	−·143	+·106	50..	−·099	−100	−·039	
25..	−·043	−·136	−·049	+·127	51..	−·014	+·042	+·036	
26..	+·165	−·065	+·040	−·021	52..	+·086	+·148	+·019	
					53..	..	..	−·046	

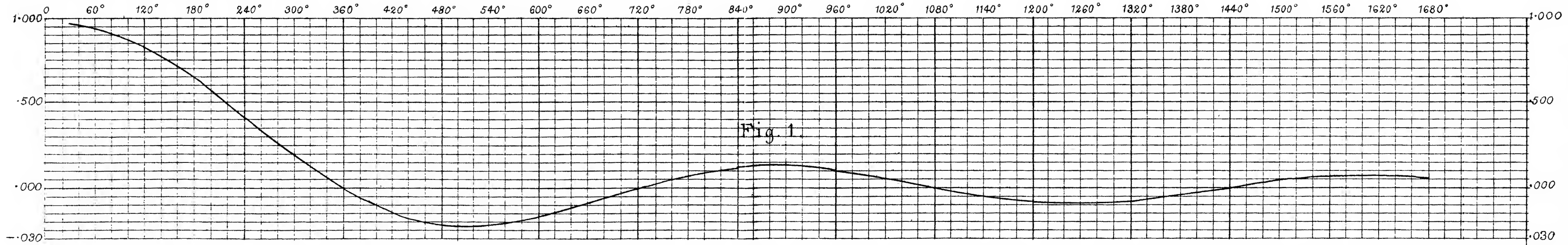
Fig. 1.

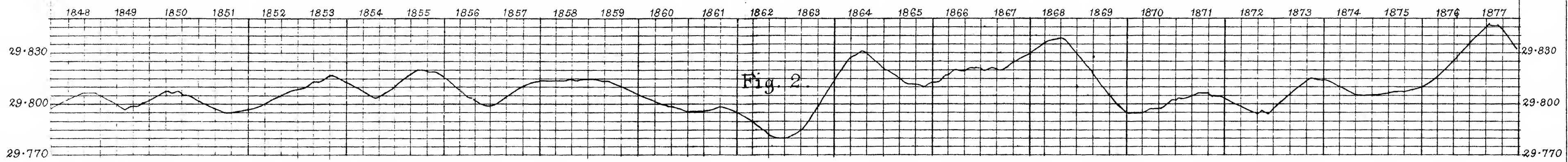
Fig. 2.

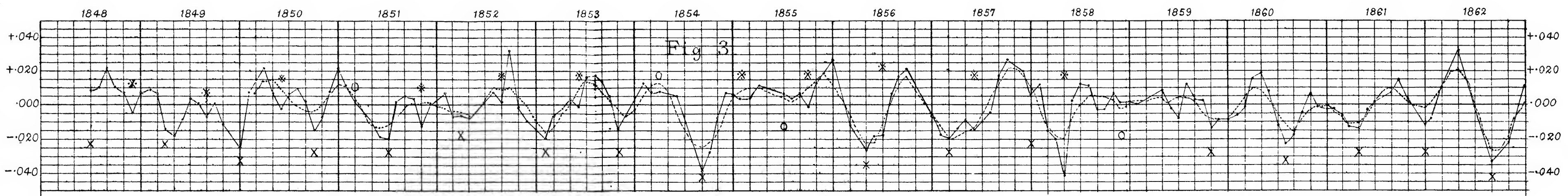
Fig. 3.

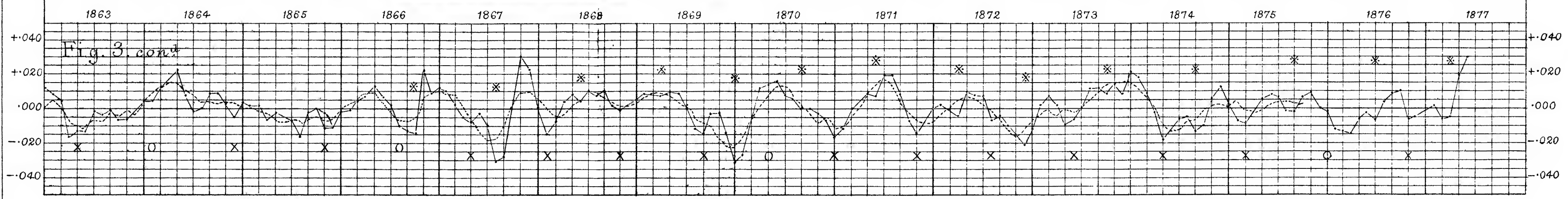
Fig. 3. con^d

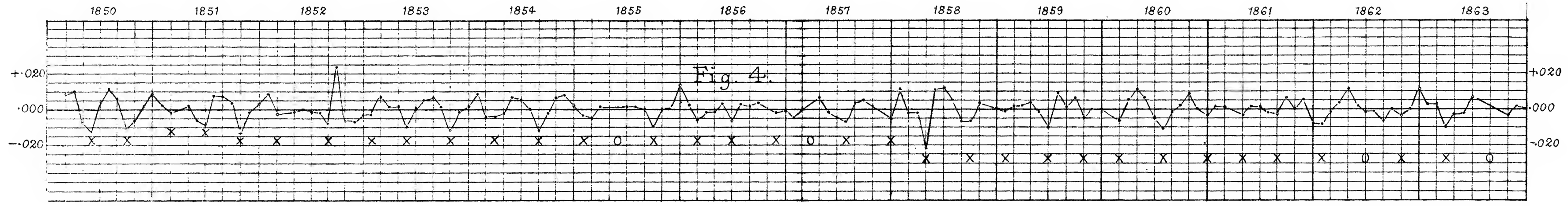
Fig. 4.

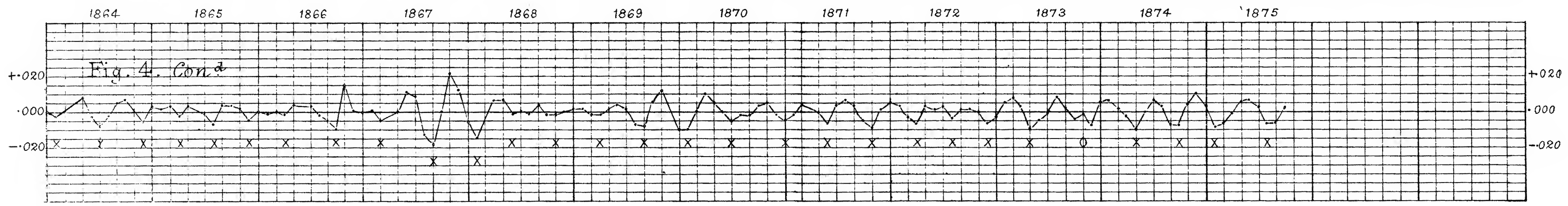
Fig. 4. Con^d

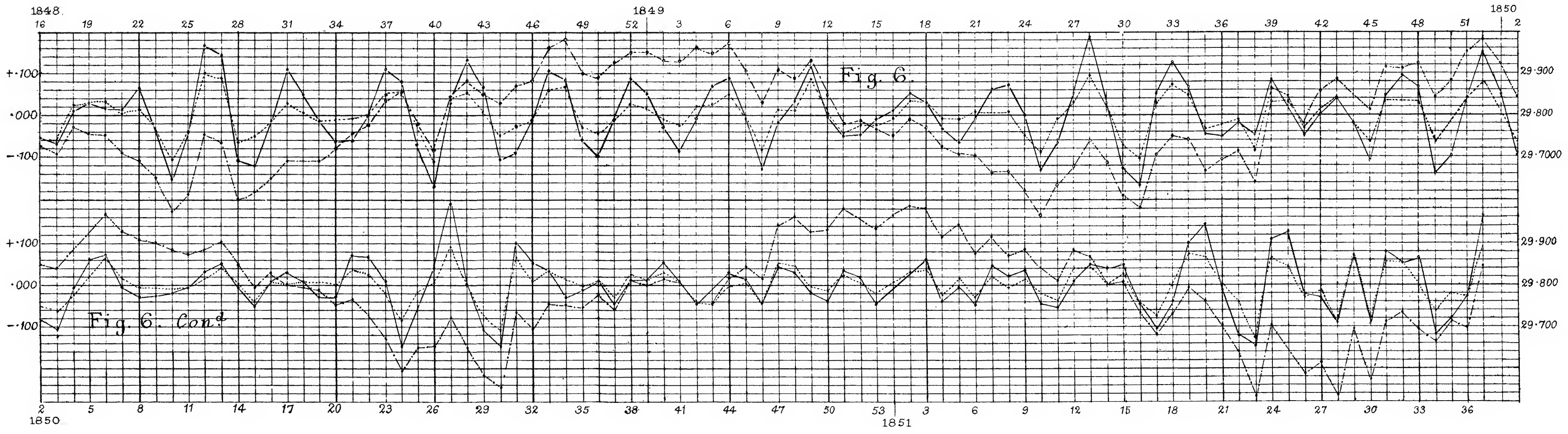
Fig. 6.

Fig. 6. Con^d

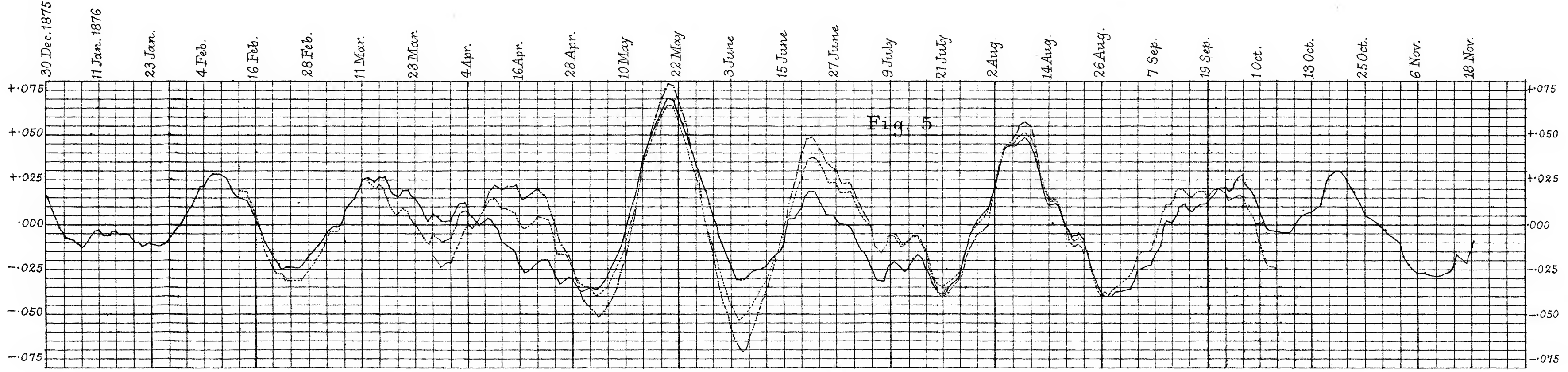
Fig. 5.

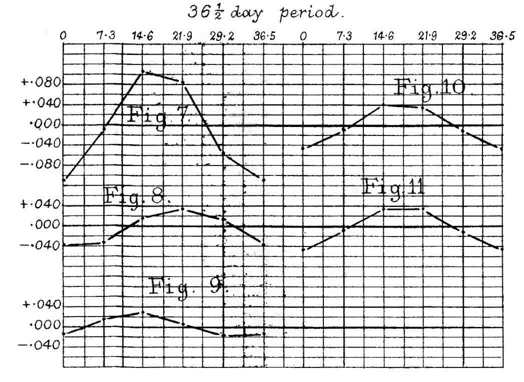
36½ day period.

Fig. 7. Fig. 8. Fig. 9. Fig. 10 Fig. 11

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